

GENERICITY OF NONDEGENERATE CRITICAL POINTS AND MORSE GEODESIC FUNCTIONALS

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ABSTRACT. We consider a family of variational problems on a Hilbert manifold parameterized by an open subset of a Banach manifold, and we discuss the genericity of the nondegeneracy condition for the critical points. Using classical techniques, we prove an abstract genericity result that employs the infinite dimensional Sard–Smale theorem, along the lines of an analogous result of B. White [27]. Applications are given by proving the genericity of metrics without degenerate geodesics between fixed endpoints in general (non compact) semi-Riemannian manifolds, in orthogonally split semi-Riemannian manifolds and in globally hyperbolic Lorentzian manifolds. We discuss the genericity property also in stationary Lorentzian manifolds.

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1. INTRODUCTION

Generic properties of flows, especially of Riemannian geodesic flows, are a classical topic in the theory of dynamical systems and in calculus of variations, with important contributions by many authors. A well known result of the area is the so-called *bumpy metric theorem*, originally formulated by Abraham [4], and proved in detail by Anosov [6], which states that the bumpy Riemannian metrics over a given compact manifold form a generic set. Recall that a metric is bumpy when all its closed geodesics are nondegenerate. Recently, B. White [27] has proven a nice formulation of the bumpy metric theorem in the context of minimal immersions; more precisely, given a compact manifold M and a

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complete Riemannian manifold (N, h) , with $\dim(M) < \dim(N)$, then the Riemannian metrics h on N such that every minimal *embedding* $\phi : (M, g) \rightarrow (N, h)$ is nondegenerate form a generic set. In the case $M = S^1$, White's theorem does not reproduce exactly the bumpy metric theorem, in that the result does not guarantee that iterates of a given closed geodesic, which are not embeddings, are also nondegenerate. A key point in the proof of this result, which has a variational nature, is that the Jacobi differential operator arising from the second variational formula of the area functional is a self-adjoint Fredholm operator. Inspired by White's result, the goal of the present paper is to initiate a study of generic properties of geodesics in *semi-Riemannian* manifolds, i.e., in manifolds endowed with a non positive definite nondegenerate metric tensor. At the present stage, this is a totally unexplored field. Motivations for the interest in such kind of dynamical systems are obviously related to Lorentzian geometry and General Relativity, to which this paper is ultimately devoted, but also to Morse theory, as explained below, and to the general theory of semi-Riemannian manifolds. As a starting point for our theory, we consider the case of *fixed endpoints* geodesics in semi-Riemannian manifolds. We set ourself the task of determining whether the set of semi-Riemannian metrics on a fixed manifold M that:

- have fixed index;
- belong to some specific class, such as orthogonally split, globally hyperbolic, or are conformal to some given metric;
- make any two arbitrarily fixed distinct points non conjugate along any geodesic,

is generic. One should observe that in the non positive definite case, the Jacobi differential operator is not self-adjoint, or even normal, but the index form (i.e., the second variation of the geodesic action functional) along a given geodesic is represented by a self-adjoint Fredholm operator. Recall that, given $p, q \in M$, the nonconjugacy property above relatively to some semi-Riemannian metric g on M is equivalent to the fact that the g -geodesic action functional $\Omega_{p,q} \ni \gamma \mapsto \frac{1}{2} \int_0^1 g(\dot{\gamma}, \dot{\gamma}) dt \in \mathbb{R}$, defined on the Hilbert manifold $\Omega_{p,q}$ of all curves of Sobolev class H^1 in M joining p and q , is a *Morse function*. Standard Morse theory does not apply to the semi-Riemannian geodesic action functional, due to the fact that in the non positive definite case all its critical points have infinite Morse index. Recent developments of Morse theory, mostly due to the work of Abbondandolo and Majer (see [1, 2]) have shown that, under suitable assumptions, one can construct a doubly infinite chain complex (Morse–Witten complex) out of the critical points of a strongly indefinite Morse functional, using the dynamics of the gradient flow. The Morse relations for the critical points are obtained by computing the homology of this complex, which in the standard Morse theory is isomorphic to the singular homology of the base manifolds. Such computation is one of the central and highly non trivial issues of the theory. Remarkably, Abbondandolo and Majer have also shown that this homology is stable by “small” perturbations, so that in several concrete examples one can reduce its computation to a simpler case. This occurs for instance in the case of the geodesic action functional in a globally hyperbolic Lorentzian manifold, in which case the homology of the Morse–Witten complex is stable by small C^0 perturbations of the metric. Thus, it becomes a relevant issue to discuss under which circumstances a given metric tensor can be perturbed in a given class in such a way that the nondegeneracy property for its geodesics between two prescribed points is preserved. This problem is the original motivation for the results developed in this paper; we basically give an affirmative answer to the genericity questions posed above, with three remarkable exceptions that will be discussed below.

The idea for proving the genericity of the nondegeneracy property for the critical points of a family of functionals, which follows a standard transversality approach (see the classical reference [5, Chapter 4], or the more recent [3, Section 2.11]), is the following. Assume that one is given a Hilbert manifold Y , and a family of functionals $f_x : Y \rightarrow \mathbb{R}$ parameterized by points x in an open subset \mathcal{A} of a Banach space X . In the geodesic case, Y is the Hilbert manifold $\Omega_{p,q}(M)$ of curves between two fixed points in a manifold M , X is the

space of $(0, 2)$ -symmetric tensors on M , and \mathcal{A} is the open set of nondegenerate tensors having a fixed index. Then, one consider the set of pairs

$$\mathfrak{M} = \{(x, y) : y \text{ is a critical point of } f_x\},$$

which under suitable assumptions has the following important properties:

- \mathfrak{M} is an embedded submanifold of the product $X \times Y$;
- the projection $\Pi : \mathfrak{M} \rightarrow X$ onto the first factor is a smooth nonlinear Fredholm map of index 0;
- the critical values of Π are precisely the set of $x \in \mathcal{A}$ such that f_x has some degenerate critical point in Y .

Thus, the genericity of nondegenerate critical points is reduced to the question of regular values of a Fredholm map, to which Sard–Smale theorem gives a complete answer. In order to make this setup working, one needs some regularity and Fredholmness assumptions, plus a certain transversality assumption that in the geodesic case reduces to the existence of some special tensors on the underlying manifold.

There are three cases in which the genericity property of nondegenerate geodesics either fails, or cannot be proven with the techniques of this paper. First, perturbations in a given conformal class are insufficient to eliminate degeneracies of *lightlike* geodesics. In fact, every conformal perturbation of a semi-Riemannian metric preserves lightlike pre-geodesics and their conjugate points, so that nondegeneracy is *not* generic in a given conformal class. The second, and more intriguing, point that deserves further attention is the case of geodesics with the same initial and endpoint and, more specifically, the case of periodic geodesics. Note that in the case of periodic geodesics, the notion of nondegeneracy has to be modified, due to the fact that in the periodic case the tangent field to a geodesic is always in the kernel of the index form. Every periodic geodesic produces a countable number of distinct critical points of the action functional by iteration. In order to develop Morse theory, one clearly needs to have nondegeneracy of all this iterates, which amounts to saying that the linearized Poincaré map along the given geodesic should not have any (complex) roots of unity in its spectrum. Due to some technical reasons, the metric perturbations studied in this paper fail to produce the desired result in the case of a 1-periodic geodesic γ some of whose iterates γ^k admits a nontrivial periodic Jacobi field J satisfying $\sum_{j=1}^k J_{t+j} = 0$ for all t . Examples of this situation can be constructed easily, for instance by considering periodic geodesics on a flat Möbius strip. Roughly speaking, the field $V_t = \sum_{j=1}^k J_{t+j}$ indicates in which direction the metric should be *stretched* in order to destroy the degeneracy produced by the Jacobi field J . Due to this problem, all our genericity results use the (probably unnecessary) assumption that the endpoints should be distinct. It is curious to observe that, also under this assumption, one does not avoid having to deal with *portions* of periodic geodesics (see Lemma 4.2), but this case is treated with a little “parity” trick. We conjecture that most of the results of this paper should hold also in the case of periodic geodesic (in the Riemannian case this is established in [4] and [6]), but the proof should be based on dynamical arguments, rather than variational. The third situation where the transversality condition is not satisfied, and thus the genericity of metrics with nondegenerate geodesics cannot be deduced by the theory in the present paper, is the case of stationary Lorentzian manifolds. We will show with an explicit example that, in the class of stationary metrics on a manifold M having a prescribed vector field $Y \in \mathfrak{X}(M)$ as timelike Killing vector field, the transversality condition fails to hold along a degenerate geodesic which is an integral line of Y .

We will now give a detailed technical description of the material discussed in the paper, with a few additional remarks. In Section 2 we fix notations and discuss a few preliminary results involving the functional analytical setup and the geometrical setup of the paper. In the functional analytical part we determine a criterion for the surjectivity (Lemma 2.1) and a criterion for existence of a closed complement to the kernel (Proposition 2.3) of the

direct sum of two bounded linear operators. These are used to determine transversality to the zero section of a cotangent bundle for the partial derivative of a map defined on product spaces (Proposition 3.1). The main result of the geometrical part is Lemma 2.4, that gives the existence of (global) sections of a vector bundle endowed with a connection, whose value and covariant derivative have been prescribed along a small immersed curve in the base. In Section 3 we consider the abstract setup of a family of smooth functionals over a Hilbert manifold, parameterized by points of (an open subset of) a Banach manifold. The central result, Corollary 3.4, uses a certain transversality assumption (see formula (3.1)) to characterize the Morse functionals in the family as regular values of a nonlinear Fredholm map, yielding the desired genericity result via Sard–Smale theorem. The main idea and the proof of Corollary 3.4 follow closely B. White’s arguments in the abstract setup of [27, Section 1]. In Section 4 we apply Corollary 3.4 to the fixed endpoint geodesic problem in several contexts. We will first consider (Subsections 4.1 and 4.2) the case of general semi-Riemannian metrics on an arbitrarily fixed manifold, possibly non compact. When dealing with a non compact manifold M , there is no canonical Banach space structure on the space of tensors on M , and in particular there is no way of describing semi-Riemannian metric tensors as an open subset of a Banach space. Note that Sard–Smale theorem uses a Banach space structure in an essential way. One way to induce a Banach space norm in the space of tensors would be to use an auxiliary complete Riemannian metric g_R on M , and then considering tensors of class C^k on M whose first k (covariant) derivatives have bounded g_R -norm (see Example 1). However, a more general genericity statement is obtained by considering the notion of C^k -Whitney type Banach space of tensors on M , which is introduced in Subsection 4.1. A Banach space of tensors \mathcal{E} is said to be of C^k -Whitney type if it contains all tensors of class C^k with compact support (these are used in all our genericity results), and if its topology is finer than the weak C^k -Whitney topology, i.e., if convergence in \mathcal{E} implies C^k -convergence on compacta. C^k -Whitney type Banach spaces of tensors seem to provide a sufficiently general and adequate environment in which one can prove genericity results based on Sard–Smale theorem, including a large variety of situations where one poses asymptotic conditions on the metric tensors. An argument by Taubes, pointed out to the authors by the referee, allows to extend all the genericity results presented in this paper to the more elegant context of the topology of C^∞ -convergence on compact subsets. In Section 5 we will discuss the details of the argument.

In Subsection 4.3 we study the genericity property of metrics in a given conformal class. As mentioned above, we restrict ourselves to the case of nondegeneracy of nonlightlike geodesics between fixed endpoints. In subsection 4.4 we consider product manifolds $M = M_1 \times M_2$, endowed with metric tensors that make the two factors orthogonal, and we prove a genericity result in this context. In Subsection 4.5 we consider globally hyperbolic Lorentzian metric tensors; by a celebrated result of Geroch ([15]), recently improved by Bernal and Sánchez ([8, 9]), these metrics form a subclass of the family of orthogonally split metric tensors in product manifolds $M_1 \times \mathbb{R}$. Finally, in Section 4.6, we will exhibit a counterexample to the transversality condition in the stationary Lorentzian case.

2. NOTATIONS AND PRELIMINARIES

2.1. Functional analytical preliminaries. Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$; given a closed subspace $W \subset H$, we will denote by $P_W : H \rightarrow W$ the orthogonal projection onto W .

Lemma 2.1. *Let V be a Banach space, H a Hilbert space, $L_1 : V \rightarrow H$ and $L_2 : H \rightarrow H$ be bounded linear operators, with $\text{Im}(L_2)$ closed; set $L = L_1 \oplus L_2 : V \oplus H \rightarrow H$, $L(v, h) = L_1(v) + L_2(h)$, $v \in V$, $h \in H$. Then, L is surjective if and only if $P_{\text{Im}(L_2)^\perp}(\text{Im}(L_1)) = \text{Im}(L_2)^\perp$. If in addition L_2 is self-adjoint and $P_{\text{Ker}(L_2)}(\text{Im}(L_1))$ is closed in $\text{Ker}(L_2)$ (this is the case, for instance, if $\text{Ker}(L_2)$ is finite dimensional, i.e., if*

L_2 is Fredholm), then L is surjective if and only if for all $h \in \text{Ker}(L_2) \setminus \{0\}$ there exists $v \in V$ such that $\langle L_1(v), h \rangle \neq 0$.

Proof. The first statement is immediate. If L_2 is self-adjoint, then $\text{Im}(L_2)^\perp = \text{Ker}(L_2)$. Since $P_{\text{Ker}(L_2)}(\text{Im}(L_1))$ is closed, $P_{\text{Ker}(L_2)} \circ L_1 : V \rightarrow \text{Ker}(L_2)$ is not surjective if and only if there exists $h \in \text{Ker}(L_2)$ such that $\langle P_{\text{Ker}(L_2)}(L_1(v)), h \rangle = \langle L_1(v), h \rangle = 0$ for all $v \in V$. The conclusion follows. \square

Let us recall that a closed subspace W of a Banach space V is said to be *complemented* if there exists a closed subspace $W' \subset V$ such that $V = W \oplus W'$; such a space W' will be called a *complement* of W in V .

Lemma 2.2. *Let $L : U \rightarrow V$ be a linear map between vector spaces, and let $S \subset V$ be a finite codimensional space. Then, $L^{-1}(S)$ is finite codimensional in U , and $\text{codim}_U(L^{-1}(S)) = \text{codim}_V(S) - \text{codim}_V(\text{Im}(L) + S)$.*

Proof. If $\pi : V \rightarrow V/S$ is the projection onto the quotient, the linear map $\pi \circ L : U \rightarrow V/S$ has kernel $L^{-1}(S)$. Hence, $\pi \circ L$ defines an injective linear map on the quotient $U/L^{-1}(S) \rightarrow V/S$, and so:

$$\begin{aligned} \text{codim}_V(S) = \dim(V/S) &= \dim(U/L^{-1}(S)) + \text{codim}_{V/S}(\text{Im}(\pi \circ L)) \\ &= \text{codim}_U(L^{-1}(S)) + \text{codim}_V(\text{Im}(L) + S). \quad \square \end{aligned}$$

Proposition 2.3. *Let U, V, W be Banach spaces, $L_1 : U \rightarrow W$, $L_2 : V \rightarrow W$ be bounded linear operators, and assume that $\text{Ker}(L_2)$ is complemented in V (this is the case, for instance, if V is a Hilbert space, or if L_2 is Fredholm) and that $\text{Im}(L_2)$ is finite codimensional in W . Set $L = L_1 \oplus L_2 : U \oplus V \rightarrow W$; then, $\text{Ker}(L)$ is complemented in $U \oplus V$.*

Proof. Consider the (possibly not closed) subspace $\text{Im}(L_1) \subset W$; $\text{Im}(L_1) \cap \text{Im}(L_2)$ has finite codimension in $\text{Im}(L_1)$. Namely, if $\pi : W \rightarrow W/\text{Im}(L_2)$ is the quotient map, then the restriction $\pi|_{\text{Im}(L_1)} : \text{Im}(L_1) \rightarrow W/\text{Im}(L_2)$ has kernel $\text{Im}(L_1) \cap \text{Im}(L_2)$. Thus, one has an injective linear map from $\text{Im}(L_1)/[\text{Im}(L_1) \cap \text{Im}(L_2)]$ to the finite dimensional space $W/\text{Im}(L_2)$, which proves our claim. Set $\text{Im}(L_1) = [\text{Im}(L_1) \cap \text{Im}(L_2)] \oplus Z$, with $Z \subset W$ a (closed) finite dimensional subspace. We now claim that $\text{Ker}(L_1)$ has finite codimension in $L_1^{-1}(Z)$; namely, one has an injective linear map from $L_1^{-1}(Z)/\text{Ker}(L_1)$ to Z . Set $L_1^{-1}(Z) = \text{Ker}(L_1) \oplus U'$, with U' a (closed) finite dimensional subspace of U . Finally, let V' be a complement of $\text{Ker}(L_2)$ in V ; we will now show that $U' \oplus V'$ is a complement of $\text{Ker}(L)$ in $U \oplus V$. Assume $(x, y) \in U' \oplus V'$ with $L_1(x) + L_2(y) = 0$; since $U' \subset L_1^{-1}(Z)$, then $L_1(x) \in Z$. But $L_1(x) = -L_2(y) \in \text{Im}(L_2)$, thus $L_1(x) \in Z \cap (\text{Im}(L_1) \cap \text{Im}(L_2)) = \{0\}$, i.e., $L_1(x) = L_2(y) = 0$. Thus, $x \in U' \cap \text{Ker}(L_1) = \{0\}$ and $y \in V' \cap \text{Ker}(L_2) = \{0\}$, which proves that $[U' \oplus V'] \cap \text{Ker}(L) = \{0\}$.

Let now $(x, y) \in U \oplus V$ be arbitrary; write $L_1(x) = L_1(u) + z$, where $u \in U$, $L_1(u) \in \text{Im}(L_2)$ and $z \in Z$. Since $z \in Z \subset \text{Im}(L_1)$, one has $z = L_1(a)$ for some $a \in U'$; thus, $x = u + a + b$ for some $b \in \text{Ker}(L_1)$. Choose $w \in V'$ such that $L_1(u) = L_2(w)$, and set $y = c + v$, where $c \in \text{Ker}(L_2)$ and $v \in V'$. Then, $(u + b, c - w) \in \text{Ker}(L)$, $(a, v + w) \in U' \oplus V'$ and $(x, y) = (u + b, c - w) + (a, v + w)$, which proves that $\text{Ker}(L) + [U' \oplus V'] = U \oplus V$. \square

2.2. Geometric preliminaries. Let M be a smooth manifold with $\dim(M) \geq 2$ and let ∇ be an arbitrarily fixed symmetric connection on TM . Given another (symmetric) connection ∇' on TM , there exists a (symmetric) $(1, 2)$ -tensor Γ on M defined by:

$$\nabla' = \nabla + \Gamma,$$

that will be called the *Christoffel tensor of ∇' relatively to ∇* . If ∇^g is the Levi-Civita connection of some semi-Riemannian metric tensor g on M , then using Koszul's formula,

its Christoffel tensor relative to ∇ is computed as follows:

$$(2.1) \quad g(\Gamma^g(X, Y), Z) = \frac{1}{2} [\nabla g(X, Z, Y) + \nabla g(Y, Z, X) - \nabla g(Z, X, Y)].$$

For all $x \in M$ and all $v \in T_x M$, we will denote by $\Gamma_x^g(v) : T_x M \rightarrow T_x M$ the map defined by $\Gamma_x^g(v)w = \Gamma_x^g(v, w)$, for all $w \in T_x M$, and by $\Gamma_x^g(v)^* : T_x M^* \rightarrow T_x M^*$ its adjoint. The curvature tensor R^g of the connection ∇^g will be chosen with the following sign convention: $R^g(X, Y) = [\nabla_X^g, \nabla_Y^g] - \nabla_{[X, Y]}^g$. The symbol \exp will denote the exponential map of the connection ∇ .

Given a smooth vector bundle $\pi : E \rightarrow M$ over M , we will denote by $\Gamma(E)$ the space of all smooth sections of E ; given a smooth map between manifolds $f : N \rightarrow M$, then $f^*(E)$ will denote the pull-back bundle over N . The fiber $\pi^{-1}(x)$ over a point $x \in M$ will be denoted by E_x ; the dimension of the typical fiber of E will be called the *rank* of E . In this paper, we will be mostly interested in *tensor bundles* over M , i.e., all those vector bundles obtained by functorial constructions from the tangent bundle TM and the cotangent bundle TM^* . Given nonnegative integers r, s , we will denote by $TM^{*(r)} \otimes TM^{(s)}$ the tensor product of r copies of TM^* and s copies of TM ; sections of $TM^{*(r)} \otimes TM^{(s)}$ are called *tensors of type (s, r)* on M .

The following is a result that says that we can find global sections of a vector bundle with prescribed value and covariant derivative along a sufficiently short curve in M .

Lemma 2.4. *Let $\pi : E \rightarrow M$ be a smooth vector bundle endowed with a connection ∇ , let $\gamma : [a, b] \rightarrow M$ be a smooth immersion and let $V \in \Gamma(\gamma^*(TM))$ be a smooth vector field along γ such that V_{t_0} is not parallel to $\dot{\gamma}(t_0)$ for some $t_0 \in]a, b[$. Then, there exists an open interval $I \subset [a, b]$ containing t_0 with the property that, given smooth sections H and K of $\gamma^*(E)$ with compact support in I and given any open set U containing $\gamma(I)$, then there exists $h \in \Gamma(E)$ with compact support contained in U , such that $h_{\gamma(t)} = H_t$ and $\nabla_{V_t} h = K_t$ for all $t \in I$.*

Proof. Let $I \subset]a, b[$ be a sufficiently small open interval such that $\gamma|_I$ is an embedding and such that V_t is not parallel to $\dot{\gamma}(t)$ for all $t \in I$; let $S \subset M$ be a smooth hypersurface containing $\gamma(I)$ and such that $V_t \notin T_{\gamma(t)} S$ for all $t \in I$. Choose a smooth section $V \in \Gamma(S^*(TM))$ such that $V(\gamma(t)) = V_t$ for all $t \in I$. By possibly reducing the size of I and S , we can assume the existence of a small positive number ε and a diffeomorphism $\phi : S \times]-\varepsilon, \varepsilon[\ni (x, \lambda) \mapsto \phi(x, \lambda) \in \tilde{U} \subset U$, where \tilde{U} is an open subset of M contained in U that contains $\gamma(I)$, such that $\frac{\partial \phi}{\partial \lambda}(x, 0) = V(x)$ for all $x \in S$. For instance, such a diffeomorphism can be constructed using the exponential map \exp' of some connection ∇' in TM by setting $\phi(x, \lambda) = \exp'_x(\lambda V(x))$ for all $(x, \lambda) \in S \times]-\varepsilon, \varepsilon[$. Clearly, \tilde{U} can be chosen small enough so that $E|_{\tilde{U}}$ admits a trivialization; let $r \in \mathbb{N}$ be the rank of E and let $p(x, \lambda) : \mathbb{R}^r \rightarrow E_{\phi(x, \lambda)}$ be a smooth referential of $\phi^*(E|_{\tilde{U}})$ with the property that $\frac{D}{d\lambda} p(x, \lambda) = 0$, i.e., p is parallel along the curves $]-\varepsilon, \varepsilon[\ni \lambda \mapsto \phi(x, \lambda)$. For instance, such referential p can be chosen by selecting an arbitrary smooth referential of E along S , and then extending by parallel transport along the curve $\lambda \mapsto \phi(x, \lambda)$. The problem of determining the required section h is now reduced to the search of a smooth map $\tilde{h} : S \times]-\varepsilon, \varepsilon[\rightarrow \mathbb{R}^r$ having compact support such that:

- $\tilde{h}(\gamma(t), 0) = p(\gamma(t), 0)^{-1} H_t$;
- $\frac{\partial \tilde{h}}{\partial \lambda}(\gamma(t), 0) = p(\gamma(t), 0)^{-1} K_t$,

for all $t \in I$. Once such \tilde{h} has been determined, the desired section h will be obtained by setting $h(\phi(x, \lambda)) = p(x, \lambda) \circ \tilde{h}(x, \lambda)$ for all $(x, \lambda) \in S \times]-\varepsilon, \varepsilon[$ and $h = 0$ outside \tilde{U} .

The function \tilde{h} can be constructed as follows. First, choose smooth maps $\tilde{H}, \tilde{K} : S \rightarrow \mathbb{R}^r$ having compact support such that $p(\gamma(t), 0) \circ \tilde{H}(\gamma(t)) = H_t$ and $p(\gamma(t), 0) \circ$

$\tilde{K}(\gamma(t)) = K_t$ for all $t \in I$. Finally, define $\tilde{h}(x, \lambda) = \tilde{H}(x) + f(\lambda)\tilde{K}(x)$, where $f :]-\varepsilon, \varepsilon[\rightarrow \mathbb{R}$ is a smooth function with compact support such that $f(\lambda) = \lambda$ near $\lambda = 0$. This concludes the construction and proves the Lemma. \square

Given a g -geodesic $\gamma : I \rightarrow M$ in M , a *Jacobi field* along γ is a smooth vector field J along γ that satisfies the second order linear equation $(\mathbf{D}^g)^2 J(t) = R^g(\dot{\gamma}(t), J(t)) \dot{\gamma}(t)$ for all t , where \mathbf{D}^g denotes covariant differentiation along γ relatively to the connection ∇^g . The endpoints of γ are said to be *conjugate* along γ if there exists a non trivial Jacobi field along γ that vanishes at both endpoints of I . Affine multiples of the tangent field $\dot{\gamma}$ are Jacobi fields; conversely, the only Jacobi fields along γ that are everywhere parallel to $\dot{\gamma}$ must be affine multiples of $\dot{\gamma}$. Other than that, Jacobi fields are parallel to the tangent field $\dot{\gamma}$ only at isolated points:

Lemma 2.5. *Let $\gamma : [a, b] \rightarrow M$ be a geodesic in (M, g) , and let J be a Jacobi field which is not everywhere parallel to $\dot{\gamma}$. Then, the set:*

$$\{t \in [a, b] : J_t \text{ is parallel to } \dot{\gamma}(t)\}$$

is finite.

Proof. Since $\dot{\gamma}$ is parallel, the covariant differentiation operator \mathbf{D}^g defines a connection on the quotient bundle $\bigcup_{t \in [a, b]} T_{\gamma(t)}M / \mathbb{R}\dot{\gamma}(t)$ over the interval $[a, b]$, that will be denoted by $\tilde{\mathbf{D}}$. Moreover, by the anti-symmetry of the curvature tensor, the linear operator $R^g(\dot{\gamma}(t), \cdot)\dot{\gamma}(t) : T_{\gamma(t)}M \rightarrow T_{\gamma(t)}M$ passes to the quotient and gives a well defined operator $\tilde{R}_t : T_{\gamma(t)}M / \mathbb{R}\dot{\gamma}(t) \rightarrow T_{\gamma(t)}M / \mathbb{R}\dot{\gamma}(t)$. Thus, the class $\tilde{J} = J + \mathbb{R}\dot{\gamma}$ satisfies the second order linear differential equation $\tilde{\mathbf{D}}^2 \tilde{J} = \tilde{R}\tilde{J}$. If the zeroes of \tilde{J} were not isolated, then \tilde{J} would be identically zero, i.e., J would be everywhere parallel to $\dot{\gamma}$. \square

3. AN ABSTRACT GENERICITY RESULT

In this section we will study the nondegeneracy of critical points of a smoothly varying family of variational problems; we will prove the result of [27, Theorem 1.2] in the context of Banach and Hilbert manifolds. The approach followed is classical (see [5, Chapter 4], or [3, Section 2.11]), and several of the results presented in this section are very likely already existing in the literature in some form. The authors have found White's formulation of the transversality assumption (see (3.1)) particularly well suited for their purposes, and decided to write complete proofs of its extension to the Banach manifold setting.

Recall that, given Banach manifolds \mathcal{X} and \mathcal{Y} , a smooth submanifold $\mathcal{Z} \subset \mathcal{Y}$, and a C^1 -map $F : \mathcal{X} \rightarrow \mathcal{Y}$, then F is said to be *transversal* to \mathcal{Z} if for all $\mathfrak{x}_0 \in F^{-1}(\mathcal{Z})$, $dF(\mathfrak{x}_0)^{-1}(T_{F(\mathfrak{x}_0)}\mathcal{Z})$ is complemented in $T_{\mathfrak{x}_0}\mathcal{X}$ and $\text{Im}(dF(\mathfrak{x}_0)) + T_{F(\mathfrak{x}_0)}\mathcal{Z} = T_{F(\mathfrak{x}_0)}\mathcal{Y}$. Under these circumstances, $\mathcal{M} = F^{-1}(\mathcal{Z})$ is a smooth embedded submanifold of \mathcal{X} , and for all $\mathfrak{x}_0 \in \mathcal{M}$, $T_{\mathfrak{x}_0}\mathcal{M}$ is given by $dF(\mathfrak{x}_0)^{-1}(T_{F(\mathfrak{x}_0)}\mathcal{Z})$.

Proposition 3.1. *Let X be a Banach manifold, Y a Hilbert manifold, and let $\mathcal{A} \subset X \times Y$ be an open subset. Assume that $f : \mathcal{A} \rightarrow \mathbb{R}$ is a map of class C^k , with $k \geq 2$, and with the property that for every $(x_0, y_0) \in \mathcal{A}$ such that $\frac{\partial f}{\partial y}(x_0, y_0) = 0$, the Hessian*

$$\frac{\partial^2 f}{\partial y^2}(x_0, y_0) : T_{y_0}Y \longrightarrow T_{y_0}Y^* \cong T_{y_0}Y$$

has finite codimensional image (i.e., $\frac{\partial^2 f}{\partial y^2}(x_0, y_0)$ is a Fredholm operator¹).

Then, the map $\frac{\partial f}{\partial y} : \mathcal{A} \rightarrow TY^$ is transversal to the zero section of TY^* if and only if for all (x_0, y_0) with $\frac{\partial f}{\partial y}(x_0, y_0) = 0$ and all $w \in \text{Ker}\left[\frac{\partial^2 f}{\partial y^2}(x_0, y_0)\right] \setminus \{0\}$ there exists*

¹Recall that the image of a bounded linear operator, if finite codimensional, is automatically closed.

$v \in T_{x_0}X$ such that

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)(v, w) \neq 0,$$

i.e.,

$$(3.1) \quad \text{Ker} \left(\frac{\partial^2 f}{\partial y^2}(x_0, y_0) \right) \cap \text{Im} \left(\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \right)^\perp = \{0\}.$$

Remark 3.2. Observe that, given y_0 , the map $x \mapsto \frac{\partial f}{\partial y}(x, y_0)$ takes values in the fixed Hilbert space $T_{y_0}Y^*$, so that the second derivative $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$ is well defined without the use of a connection on TY^* . Similarly, the second derivative $\frac{\partial^2 f}{\partial y^2}(x_0, y_0)$ is well defined when $\frac{\partial f}{\partial y}(x_0, y_0) = 0$, and it is the Hessian of the function $y \mapsto f(x_0, y)$ at the critical point y_0 .

Proof. Denote by $\mathbf{0}$ the zero section of TY^* . For all $y \in Y$, denoting by 0_y the zero in T_yY^* , the tangent space $T_{0_y}\mathbf{0}$ is identified canonically with T_yY , so that $T_{0_y}TY^* \cong T_yY \oplus T_yY^*$; let $\pi_y : T_{0_y}TY^* \rightarrow T_yY^*$ denote the projection relative to this decomposition. Given $(x_0, y_0) \in \mathcal{A}$ with $\frac{\partial f}{\partial y}(x_0, y_0) = 0$, the composition

$$\pi_{y_0} \circ d \left(\frac{\partial f}{\partial y} \right) (x_0, y_0) : T_{x_0}X \oplus T_{y_0}Y \longrightarrow T_{y_0}Y^*$$

is given by the direct sum of the bounded operators:

$$L_1 := \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) : T_{x_0}X \longrightarrow T_{y_0}Y^* \cong T_{y_0}Y$$

and

$$L_2 := \frac{\partial^2 f}{\partial y^2}(x_0, y_0) : T_{y_0}Y \longrightarrow T_{y_0}Y^* \cong T_{y_0}Y.$$

Transversality of $\frac{\partial f}{\partial y}$ to the zero section of TY^* is equivalent to $\text{Ker}(L_1 \oplus L_2)$ being complemented in $T_{x_0}X \oplus T_{y_0}Y$ and $L_1 \oplus L_2$ being surjective. The condition that $\text{Ker}(L_1 \oplus L_2)$ is complemented in $T_{x_0}X \oplus T_{y_0}Y$ follows immediately from Proposition 2.3, which uses our assumptions on the Hessian $\frac{\partial^2 f}{\partial y^2}(x_0, y_0)$. By Lemma 2.1, using the fact that L_2 is self-adjoint, the surjectivity of $L_1 \oplus L_2$ is equivalent to our final assumption on the mixed second derivative $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$. This concludes the proof. \square

Corollary 3.3. *In the hypotheses of Proposition 3.1, assume that the transversality condition (3.1) is satisfied at every point (x_0, y_0) with $\frac{\partial f}{\partial y}(x_0, y_0) = 0$. Then, the set:*

$$\mathfrak{M} = \left\{ (x, y) \in \mathcal{A} : \frac{\partial f}{\partial y}(x, y) = 0 \right\}$$

is an embedded C^{k-1} -submanifold of $X \times Y$. For $(x_0, y_0) \in \mathfrak{M}$, the tangent space $T_{(x_0, y_0)}\mathfrak{M}$ is given by:

$$(3.2) \quad T_{(x_0, y_0)}\mathfrak{M} = \left\{ (v, w) \in T_{x_0}X \oplus T_{y_0}Y : \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)v + \frac{\partial^2 f}{\partial y^2}(x_0, y_0)w = 0 \right\}. \quad \square$$

Let us recall that a Morse function on a Hilbert manifold is a smooth map all of whose critical points are (strongly) nondegenerate. A subset of a metric space is said to be *generic* if it is the countable intersection of dense open subsets; by Baire's theorem, a generic set is dense.

Corollary 3.4. *Under the assumptions of Corollary 3.3, if $\Pi : X \times Y \rightarrow X$ is the projection onto the first factor, then the restriction of Π to \mathfrak{M} is a nonlinear C^{k-1} Fredholm*

map of index zero. The critical points of $\Pi|_{\mathfrak{M}}$ are elements $(x_0, y_0) \in \mathfrak{M}$ such that y_0 is a degenerate critical point of the functional $\mathcal{A}_{x_0} \ni y \mapsto f(x_0, y) \in \mathbb{R}$, where

$$\mathcal{A}_x = \{y \in Y : (x, y) \in \mathcal{A}\}.$$

If X and Y are separable, then the set of $x \in X$ such that the functional $\mathcal{A}_x \ni y \mapsto f(x, y) \in \mathbb{R}$ is a Morse function is generic in the open set $\Pi(\mathcal{A}) \subset X$.

Proof. Fix $(x_0, y_0) \in \mathfrak{M}$. The kernel of $d\Pi(x_0, y_0)|_{T_{(x_0, y_0)}\mathfrak{M}}$ is given by $T_{(x_0, y_0)}\mathfrak{M} \cap [\{0\} \oplus T_{y_0}Y]$. This space is (isomorphic to) $\text{Ker}\left(\frac{\partial^2 f}{\partial y^2}(x_0, y_0)\right)$, which is finite dimensional. From (3.2), the image $d\Pi(x_0, y_0)(T_{(x_0, y_0)}\mathfrak{M})$ is given by the inverse image

$$\left[\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)\right]^{-1} \left[\text{Im}\left(\frac{\partial^2 f}{\partial y^2}(x_0, y_0)\right)\right];$$

since $\frac{\partial^2 f}{\partial y^2}(x_0, y_0)$ is Fredholm, its image has finite codimension in $T_{y_0}Y$. By Lemma 2.2, also $(T_{(x_0, y_0)}\mathfrak{M})$ has finite codimension in $T_{x_0}X$, so that $d\Pi(x_0, y_0)(T_{(x_0, y_0)}\mathfrak{M})$ is closed and therefore Fredholm. In fact, since by assumption (3.1) the linear map $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \oplus \frac{\partial^2 f}{\partial y^2}(x_0, y_0)$ is surjective, we have that by Lemma 2.2 the codimension of the image of $d\Pi(x_0, y_0)|_{T_{(x_0, y_0)}\mathfrak{M}}$ equals the codimension of $\text{Im}\left(\frac{\partial^2 f}{\partial y^2}(x_0, y_0)\right)$; as $\frac{\partial^2 f}{\partial y^2}(x_0, y_0)$ is a self-adjoint Fredholm operator, this codimension coincides with the dimension of

$$\text{Im}\left(\frac{\partial^2 f}{\partial^2 y}(x_0, y_0)\right)^\perp = \text{Ker}\left(\frac{\partial^2 f}{\partial^2 y}(x_0, y_0)\right),$$

so that the Fredholm index of $d\Pi(x_0, y_0)|_{T_{(x_0, y_0)}\mathfrak{M}}$ is equal to zero.

It is easily seen that (x_0, y_0) is a regular point of $\Pi|_{\mathfrak{M}}$, i.e., that $d\Pi(x_0, y_0)|_{T_{(x_0, y_0)}\mathfrak{M}}$ is surjective, if and only if:

$$(3.3) \quad \text{Im}\left(\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)\right) \subset \text{Im}\left(\frac{\partial^2 f}{\partial y^2}(x_0, y_0)\right);$$

using again that $\text{Im}\left(\frac{\partial^2 f}{\partial^2 y}(x_0, y_0)\right)^\perp = \text{Ker}\left(\frac{\partial^2 f}{\partial^2 y}(x_0, y_0)\right)$, and taking orthogonal complements, (3.3) becomes:

$$\text{Im}\left(\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)\right)^\perp \supset \text{Ker}\left(\frac{\partial^2 f}{\partial y^2}(x_0, y_0)\right).$$

Using assumption (3.1), (x_0, y_0) is a regular point of $\Pi|_{\mathfrak{M}}$ if and only if $\text{Ker}\left(\frac{\partial^2 f}{\partial y^2}(x_0, y_0)\right)$ is trivial, i.e., if and only if x_0 is a nondegenerate critical point of $x \mapsto f(x, y_0)$.

Thus, the set of $x \in X$ such that the functional $\mathcal{A}_x \ni y \mapsto f(x, y) \in \mathbb{R}$ is a Morse function coincides with the set of regular values of the map $\Pi|_{\mathfrak{M}}$. The last statement follows now immediately from Corollary 3.3 and Sard–Smale’s theorem (see [26]). \square

Remark 3.5. We will apply Corollary 3.4 in situations where the Banach manifold X is indeed an open subset of a Banach space E . In this case, the partial derivative $\frac{\partial f}{\partial x}$ is a map on $X \times Y$ taking value in the fixed Banach space E^* , and thus it can be differentiated with respect to the second variable y . Given $(x_0, y_0) \in \mathfrak{M}$, we have two maps:

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) : E \rightarrow T_{y_0}Y^*, \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) : T_{y_0}Y \rightarrow E^*.$$

Using local charts and Schwarz Lemma, it is easy to see that these two maps are transpose of each other. In particular, if we consider $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$ as a bilinear form on $E \times T_{y_0}Y$

and $\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$ as a bilinear form on $T_{y_0}Y \times E$, then:

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)[v, w] = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)[w, v], \quad \forall v \in E, w \in T_{y_0}Y.$$

4. MORSE GEODESIC FUNCTIONALS

4.1. Semi-Riemannian metrics. Let us consider a smooth² manifold M with $\dim(M) = n$. Given $k \geq 2$ and $\nu \in \{0, \dots, n\}$, we will denote by $\text{Met}_\nu^k(M)$ the set of all metric tensors g on M of class C^k and having index ν . This is a subset of the vector space $\Gamma_{\text{sym}}^k(TM^* \otimes TM^*)$ of all sections \mathfrak{b} of class C^k of the vector bundle $TM^* \otimes TM^*$ such that $\mathfrak{b}_x : T_x M \times T_x M \rightarrow \mathbb{R}$ is symmetric for all x .

It will be interesting to consider the case of *non compact* manifolds M , in which case there is no canonical Banach structure on the space of tensors over M . In order to overcome this problem, it will be useful to consider the following definition. A vector subspace \mathcal{E} of $\Gamma_{\text{sym}}^k(TM^* \otimes TM^*)$ will be called³ a C^k -Whitney type Banach space of tensor fields over M when:

- (a) \mathcal{E} contains all tensor fields in $\Gamma_{\text{sym}}^k(TM^* \otimes TM^*)$ having compact support;
- (b) \mathcal{E} is endowed with a Banach space norm $\|\cdot\|_{\mathcal{E}}$ with the property that $\|\cdot\|_{\mathcal{E}}$ -convergence of a sequence implies convergence in the weak Whitney C^k -topology.

More explicitly, axiom (b) above means that given any sequence $(\mathfrak{b}_n)_{n \in \mathbb{N}}$ and an element \mathfrak{b}_∞ in \mathcal{E} such that $\lim_{n \rightarrow \infty} \|\mathfrak{b}_n - \mathfrak{b}_\infty\|_{\mathcal{E}} = 0$, and given any compact subset $K \subset M$, then the restriction $\mathfrak{b}_n|_K$ tends to $\mathfrak{b}_\infty|_K$ in the C^k -topology as $n \rightarrow \infty$.

1. Example. Examples of C^k -Whitney type Banach spaces of tensor fields over M can be obtained easily introducing an auxiliary Riemannian metric g_R on M , whose Levi-Civita connection will be denoted by $\overline{\nabla}$. The choice of the Riemannian metric g_R induces naturally a connection on all vector bundles over M that are obtained by functorial constructions from the tangent bundle TM . Moreover, for all $r, s \in \mathbb{N}$, we have Hilbert space norms on every tensor product $T_x M^{*(r)} \otimes T_x M^{(s)}$ induced by g_R ; all these norms will be denoted by the same symbol $\|\cdot\|_R$. Then, we will denote by $\Gamma_{\text{sym}}^k(TM^* \otimes TM^*; g_R)$ the subset of $\Gamma_{\text{sym}}^k(TM^* \otimes TM^*)$ consisting of all section \mathfrak{b} such that:

$$(4.1) \quad \|\mathfrak{b}\|_k = \max_{j=0, \dots, k} \left[\sup_{x \in M} \left\| \overline{\nabla}^j \mathfrak{b}(x) \right\|_R \right] < +\infty.$$

When M is compact $\Gamma_{\text{sym}}^k(TM^* \otimes TM^*; g_R) = \Gamma_{\text{sym}}^k(TM^* \otimes TM^*)$. Endowed with the norm $\|\cdot\|_k$ in (4.1), $\Gamma_{\text{sym}}^k(TM^* \otimes TM^*; g_R)$ is a separable normed space, which is complete provided that the Riemannian metric g_R is chosen to be complete. Clearly, $\Gamma_{\text{sym}}^k(TM^* \otimes TM^*; g_R)$ contains all elements in $\Gamma_{\text{sym}}^k(TM^* \otimes TM^*)$ having compact support. Moreover, $\|\cdot\|_k$ -convergence implies C^k -convergence on compact sets. Thus, $\Gamma_{\text{sym}}^k(TM^* \otimes TM^*; g_R)$ is an example of C^k -Whitney type Banach spaces of tensor fields over M .

Other examples of C^k -Whitney type Banach spaces of tensor fields over M can be obtained by considering elements in $\Gamma_{\text{sym}}^k(TM^* \otimes TM^*)$ satisfying suitable boundedness assumptions at infinity on the first k covariant derivatives. Asymptotic flatness is a typical assumption, particularly fashionable among physicists.

²For the remainder of the article, we will be somewhat sloppy about the use of the adjective “smooth”. In the case of manifolds, by smooth we will always mean “of class C^k , with $k \geq 3$ ”, and in the case of tensors, in particular metric tensors, smooth will mean “of class C^k , with $k \geq 2$ ”. This guarantees that the corresponding geodesic action functionals are of class at least C^2 . Clearly, manifolds are to be of class strictly larger than the required regularity class of tensors.

³In this paper we will only be interested in metric tensor fields, but clearly a similar definition may be given for tensor fields of all kind over M .

In the statements of some of our results, we will consider open subsets \mathcal{A} of a given C^k -Whitney type Banach space \mathcal{E} of tensor fields over M , where the elements of \mathcal{A} are assumed to be semi-Riemannian metric tensors of a given index. It is easy to show that, when M is not compact, the set $\text{Met}_\nu^k(M) \cap \Gamma_{\text{sym}}^k(TM^* \otimes TM^*; g_R)$ is *not* open in $\Gamma_{\text{sym}}^k(TM^* \otimes TM^*; g_R)$. A typical open⁴ subset of $\Gamma_{\text{sym}}^k(TM^* \otimes TM^*; g_R)$ consisting of semi-Riemannian metric tensors of index ν is:

$$\text{Met}_{\nu,*}^k(M; g_R) = \left\{ \mathbf{b} \in \text{Met}_\nu^k(M) \cap \Gamma_{\text{sym}}^k(TM^* \otimes TM^*; g_R) : \sup_{x \in M} \|\mathbf{b}_x^{-1}\|_{\mathbb{R}} < +\infty \right\};$$

here, \mathbf{b}_x^{-1} is the inverse of \mathbf{b}_x seen as a linear operator $\mathbf{b}_x : T_x M \rightarrow T_x M^*$. The assumption $\sup_{x \in M} \|\mathbf{b}_x^{-1}\|_{\mathbb{R}} < +\infty$ is equivalent to requiring that the eigenvalue with minimum

absolute value of the g_R -symmetric operator \mathbf{b}_x stays away from 0 uniformly on M .

Let $p, q \in M$ be fixed points, and let $\Omega_{p,q}(M)$ denote the set of all curves $\gamma : [0, 1] \rightarrow M$ of Sobolev class H^1 such that $\gamma(0) = p$ and $\gamma(1) = q$; it is well known that $\Omega_{p,q}(M)$ is endowed with a Hilbert manifold structure modeled on the separable Hilbert space $H_0^1([0, 1], \mathbb{R}^n)$. For $\gamma \in \Omega_{p,q}(M)$, the pull-back bundle $\gamma^*(TM)$ is endowed with a Riemannian structure on the fibers induced by the Riemannian structure g_R . The tangent space $T_\gamma \Omega_{p,q}(M)$ is identified with the Hilbert space of all sections V of $\gamma^*(TM)$ having Sobolev class H^1 , and satisfying $V(0) = V(1) = 0$. For the purposes of this paper, the choice of a specific Hilbert–Riemann structure on the infinite dimensional manifold $\Omega_{p,q}(M)$ will not be relevant; however, it will be useful to have at disposal the following inner product on the tangent spaces $T_\gamma \Omega_{p,q}(M)$:

$$(4.2) \quad \langle V, W \rangle = \int_0^1 g_R(\mathbf{D}^R V, \mathbf{D}^R W) dt.$$

Here, g_R is an arbitrarily fixed complete Riemannian metric on M and \mathbf{D}^R denotes covariant differentiation of vector fields along γ with respect to the Levi–Civita connection of g_R .

4.2. Genericity of metrics without degenerate geodesics. We will henceforth consider a fixed C^k -Whitney type Banach space \mathcal{E} of tensor fields over M and a (non empty) open subset \mathcal{A} of \mathcal{E} with $\mathcal{A} \subset \mathcal{E} \cap \text{Met}_\nu^k(M)$. A complete Riemannian metric g_R is also assumed to be fixed, in order to use the Hilbert manifold structure (4.2) in $\Omega_{p,q}(M)$. Consider the geodesic action functional:

$$F : \mathcal{A} \times \Omega_{p,q}(M) \longrightarrow \mathbb{R}$$

defined by:

$$F(g, \gamma) = \frac{1}{2} \int_0^1 g(\dot{\gamma}, \dot{\gamma}) dt.$$

This is a map of class C^k . More precisely, F is smooth (i.e., C^∞) in the variable $g \in \mathcal{A}$, while in the variable γ it is of class C^k , the same regularity required for the metrics. This is easily proved, observing that taking j derivatives of F with respect to the variable γ involves⁵ the first j derivatives of the metric g .

⁴In order to see that the set $\text{Met}_{\nu,*}^k(M; g_R)$ is open in $\Gamma_{\text{sym}}^k(TM^* \otimes TM^*; g_R)$, one uses the fact that the function $A \mapsto \lambda_*(A) = \min \{ |\lambda| : \lambda \text{ is an eigenvalue of } A \}$ is Lipschitz continuous on the set of symmetric operators A on \mathbb{R}^n . This is proved easily using the equality $\lambda_*(A) = \min_{\|x\|=1} \|Ax\|$, from which one deduces that $|\lambda_*(A) - \lambda_*(B)| \leq \|A - B\|$ for all symmetric operators A and B .

⁵For instance, the first derivative $\frac{\partial F}{\partial \gamma}(\gamma_0, g_0)$ in the direction $V \in T_\gamma \Omega_{p,q}(M)$ is given by the integral $\int_0^1 g_0(\dot{\gamma}_0, \mathbf{D}^{g_0} V) dt$, where \mathbf{D}^{g_0} is the covariant derivative of vector fields along γ relatively to the Levi–Civita connection ∇^{g_0} of g_0 . This requires the Christoffel tensors of g , which are computed in terms of the first derivatives of the metric coefficients. The second derivative $\frac{\partial^2 F}{\partial \gamma^2}(\gamma_0, g_0)$ involves the curvature tensor of ∇^{g_0} (see formula (4.3)), i.e., the second derivative of g . Higher order derivatives of F with respect to γ are computed in terms of higher order covariant derivatives of the curvature tensor of ∇^{g_0} .

Given $g_0 \in \mathcal{A}$ and $\gamma_0 \in \Omega_{p,q}(M)$, then $\frac{\partial F}{\partial \gamma}(g_0, \gamma_0) = 0$ if and only if γ_0 is a g_0 -geodesic⁶ in M joining p and q . Given one such pair (g_0, γ_0) , the second derivative $\frac{\partial^2 F}{\partial \gamma^2}$ at (g_0, γ_0) is:

$$(4.3) \quad \frac{\partial^2 F}{\partial \gamma^2}(g_0, \gamma_0)(V, W) = \int_0^1 g_0(\mathbf{D}^{g_0} V, \mathbf{D}^{g_0} W) + g_0(R^{g_0}(\dot{\gamma}_0, V) \dot{\gamma}_0, W) dt,$$

where \mathbf{D}^{g_0} denotes the covariant derivative along γ_0 induced by the Levi-Civita connection ∇^{g_0} of g_0 , and R^{g_0} is the curvature tensor of ∇^{g_0} . This is the classical *index form* of γ_0 relatively to the metric g_0 .

Lemma 4.1. $\frac{\partial^2 F}{\partial \gamma^2}(g_0, \gamma_0)$ is a Fredholm symmetric bilinear form on $T_{\gamma_0} \Omega_{p,q}(M)$, i.e., it is represented by a self-adjoint Fredholm operator on $T_{\gamma_0} \Omega_{p,q}(M)$ relatively to the inner product (4.2).

Proof. For all $t \in [0, 1]$, let $A_t : T_{\gamma(t)} M \rightarrow T_{\gamma(t)} M$ be the g_R -symmetric automorphism such that $g_0 = g_R(A_t \cdot, \cdot)$ on $T_{\gamma(t)} M$. The map $\Phi : T_{\gamma_0} \Omega_{p,q}(M) \ni V \mapsto \tilde{V} \in T_{\gamma_0} \Omega_{p,q}(M)$ defined by $\tilde{V}(t) = A_t V(t)$ is an isomorphism; we will show that $\frac{\partial^2 F}{\partial \gamma^2}(g_0, \gamma_0)$ is represented relatively to the inner product (4.2) by an operator which is a compact perturbation of Φ . Namely, the difference $E(V, W) = \frac{\partial^2 F}{\partial \gamma^2}(g_0, \gamma_0)(V, W) - \langle \Phi V, W \rangle$ is easily computed as:

$$E(V, W) = \int_0^1 \left[-g_R(A'V, \mathbf{D}^R W) + g_R(A\Gamma^R V, \mathbf{D}^R W) + g_R(A\mathbf{D}^R V, \Gamma^R W) + g_R(A\Gamma^R V, \Gamma^R W) + g_R(ARV, W) \right] dt,$$

where $\Gamma^R = \mathbf{D}^{g_0} - \mathbf{D}^R$ is the Christoffel tensor of ∇^{g_0} relatively to ∇^R . Each term in the right hand side of the above equality is bilinear in (V, W) , and does not contain any derivative of at least one of its two arguments, i.e., it is continuous relatively to the C^0 -topology in one of its arguments. From the compactness of the inclusion $H^1 \hookrightarrow C^0$, it follows easily that E is represented by a compact operator on $T_{\gamma_0} \Omega_{p,q}(M)$. \square

The kernel of the index form $\frac{\partial^2 F}{\partial \gamma^2}(g_0, \gamma_0)$ is the space of all Jacobi fields J along γ_0 such that $J(0) = J(1) = 0$. The second mixed derivative $\frac{\partial^2 F}{\partial g \partial \gamma}$ is computed as follows; let $]-\varepsilon, \varepsilon[\ni s \mapsto g_s \in \mathcal{A}$ be a smooth variation of g_0 , with $\frac{d}{ds}|_{s=0} g_s = h \in \mathcal{E}$. As we have seen in Subsection 2.2, in order to perform this computation we will fix an arbitrary symmetric connection ∇ on M ; we will make a specific choice of such connection when needed (see proof of Proposition 4.3). Using the Christoffel tensor Γ^{g_s} of the metric g_s relatively to ∇ (see (2.1)), we compute:

$$\begin{aligned} (4.4) \quad \frac{\partial^2 F}{\partial g \partial \gamma}(g_0, \gamma_0)(h, V) &= \frac{d}{ds} \Big|_{s=0} \int_0^1 g_s(\dot{\gamma}_0, \mathbf{D}^{g_s} V) dt \\ &= \frac{d}{ds} \Big|_{s=0} \int_0^1 g_s(\dot{\gamma}_0, \mathbf{D} V) + g_s(\dot{\gamma}, \Gamma^{g_s}(\dot{\gamma}, V)) dt \\ &= \int_0^1 h(\dot{\gamma}_0, \mathbf{D} V) dt + \frac{1}{2} \frac{d}{ds} \Big|_{s=0} \int_0^1 \nabla g_s(V, \dot{\gamma}_0, \dot{\gamma}_0) + \nabla g_s(\dot{\gamma}_0, \dot{\gamma}_0, V) - \nabla g_s(\dot{\gamma}_0, V, \dot{\gamma}_0) dt \\ &= \int_0^1 h(\dot{\gamma}_0, \mathbf{D} V) + \frac{1}{2} \nabla h(V, \dot{\gamma}_0, \dot{\gamma}_0) dt. \end{aligned}$$

We will need to study the self intersections of geodesics, and the following elementary result will be useful:

⁶By geodesic, we will always mean an *affinely parameterized* geodesic.

Lemma 4.2. *Let (M, g) be a semi-Riemannian manifold, and let $\gamma : [0, 1] \rightarrow M$ be a geodesic. Then, the set:*

$$\{(s, t) \in [0, 1] \times [0, 1] : s \neq t, \gamma(s) = \gamma(t)\}$$

is finite, unless γ is a closed geodesic with period $T < 1$.

Proof. Assume the existence of sequences s_n and t_n in $[0, 1]$, with $s_n \neq t_n$, $\gamma(s_n) = \gamma(t_n)$ and $s_i \neq s_j$ for all $i \neq j$ (otherwise the pairs (s_n, t_n) would be a finite number). Because of the local injectivity of γ we can assume that $t_i \neq t_j$ for all $i \neq j$, and up to taking subsequences, that $\lim s_n = s$ and $\lim t_n = t$, with $s, t \in [0, 1]$; we can also assume that $s_n \neq s$ and $t_n \neq t$ for all n . Clearly, $\gamma(s) = \gamma(t)$; since γ is locally injective (it is an immersion), then it must be $s \neq t$, say $t > s$. Set $\mu(r) = \gamma(r - t + s)$; this is a geodesic, defined for r in a neighborhood of t , and such that $\mu(t) = \gamma(t)$. Moreover, set $t'_n = s_n - s + t$; this is a sequence converging to t , and with $t'_n \neq t$ for all n . We have $\mu(t'_n) = \gamma(t_n)$ for all n , and this implies that the tangent vectors $\dot{\mu}(t) = \dot{\gamma}(s)$ and $\dot{\gamma}(t)$ are linearly dependent. Since γ is affinely parameterized, it must be $\dot{\gamma}(s) = \dot{\gamma}(t)$, which implies that γ is periodic with period $T = t - s \leq 1$. It can't be $T = 1$, i.e., $s = 0$ and $t = 1$, because otherwise it would be $\gamma(t_n) = \gamma(s_n) = \gamma(s_n + 1)$ for all n , with $t_n < 1$ and $s_n + 1 > 1$ converging to 1, contradicting the local injectivity of γ around 1. \square

Proposition 4.3. *Let M be a smooth manifold, let $\mathcal{E} \subset \Gamma_{\text{sym}}^k(TM^* \otimes TM^*)$ be a C^k -Whitney type Banach space of tensors over M , with $k \geq 2$, let $\nu \in \{0, \dots, \dim(M)\}$ be fixed and let $\mathcal{A} \subset \mathcal{E} \cap \text{Met}_\nu^k(M)$ be an open subset of \mathcal{E} . Given any pair of distinct points $p, q \in M$, the set of semi-Riemannian metrics $g \in \mathcal{A}$ such that all g -geodesics joining p and q are nondegenerate, is generic in \mathcal{A} .*

Proof. We will prove the result as application of Corollary 3.4 to the geodesic setup above. In view of the Fredholmness result of Lemma 4.1, we only need to check that the transversality condition (3.1) is satisfied in this context. We need to prove that, given a semi-Riemannian metric $g_0 \in \mathcal{A}$, a g_0 -geodesic γ_0 joining p and q , and a non trivial g_0 -Jacobi field V along γ_0 , with $V_0 = V_1 = 0$, then there exists $h \in \mathcal{E}$ for which the quantity in the last term of (4.4) does not vanish. We will find such an h to be a symmetric $(0, 2)$ -tensor of class C^k having compact support in M , and thus $h \in \mathcal{E}$. Assume first that γ_0 is not a portion of a closed geodesic in M with minimal period $T < 1$. Then, by Lemma 4.2, γ_0 has at most a finite number of self-intersections. We can therefore find an open subinterval $I \subset [0, 1]$ with the following properties:

- (a) $t \in I$ and $s \notin I$ implies $\gamma_0(s) \neq \gamma_0(t)$;
- (b) V_t is not parallel to $\dot{\gamma}_0(t)$ for all $t \in I$.

As to property (b), observe that since V is a nontrivial Jacobi field which vanishes at the endpoints, then it is not everywhere multiple of $\dot{\gamma}_0$, and by Lemma 2.5 the set of instants t at which V_t is parallel to $\dot{\gamma}_0(t)$ is finite. Choose now an open subset $U \subset M$ containing $\gamma_0(I)$ and such that

$$(4.5) \quad \gamma_0(t) \in U \cap \gamma_0([0, 1]) \iff t \in I;$$

for instance, take U to be the complement of the compact set $\gamma_0([0, 1] \setminus I)$. We will now use the result of Lemma 2.4 applied to the case of symmetric $(0, 2)$ -tensor fields, as follows. For $t \in I$, we choose H_t identically zero, and K_t a symmetric bilinear form on $T_{\gamma_0(t)}M$ (depending smoothly on t) such that $K_t(\dot{\gamma}_0(t), \dot{\gamma}_0(t)) \geq 0$ with $\int_I K_t(\dot{\gamma}_0(t), \dot{\gamma}_0(t)) dt > 0$. By possibly reducing the size of the interval I , we can assume that the thesis of Lemma 2.4 applies, and we get a globally defined smooth symmetric $(2, 0)$ -tensor h on M , having compact support contained in U , such that $h_{\gamma_0(t)} = 0$ and $\nabla_{V_t} h = K_t$ for all $t \in I$. For such h , by (4.5) we have:

$$\int_0^1 \left[h(\dot{\gamma}_0, \mathbf{D}V) + \frac{1}{2} \nabla h(V, \dot{\gamma}_0, \dot{\gamma}_0) \right] dt = \frac{1}{2} \int_I K_t(\dot{\gamma}_0(t), \dot{\gamma}_0(t)) dt > 0,$$

which concludes the proof when γ_0 is not periodic of period $T < 1$.

Assume now that γ_0 is periodic, of period $T < 1$. Consider the following numbers:

$$t_* = \min \{t > 0 : \gamma_0(t) = q\}, \quad k_* = \max \{k \in \mathbb{Z} : kT < 1\},$$

for which the following hold:⁷

$$k_* \geq 1, \quad 0 < t_* < T, \quad 1 = k_*T + t_*.$$

The geodesics $\gamma_1 = \gamma_0|_{[0, t_*]}$ and $\gamma_2 = \gamma_0|_{[t_*, T]}$ join p and q (γ_2 with the opposite orientation), and the first part of the proof applies to both γ_1 and γ_2 . Thus, we can find open intervals $I_1 = [a_1, b_1] \subset [0, t_*]$ and $I_2 = [a_2, b_2] \subset [t_*, T]$ such that:

- (a1) $t \in I_1, s \in ([0, t_*] \setminus I_1) \cup [t_*, T]$ implies $\gamma_0(s) \neq \gamma_0(t)$;
- (a2) $t \in I_2, s \in ([t_*, T] \setminus I_2) \cup [0, t_*]$ implies $\gamma_0(s) \neq \gamma_0(t)$.

We can also find open subsets $U_1, U_2 \subset M$, with $\gamma(I_i) \subset U_i, i = 1, 2$, satisfying:

$$(4.6) \quad \begin{aligned} \gamma_0(t) \in U_1 \cap \gamma_0(I_1) &\iff \exists r \in \{0, \dots, k_*\} \text{ such that } t - rT \in I_1, \\ \gamma_0(t) \in U_2 \cap \gamma_0(I_2) &\iff \exists r \in \{0, \dots, k_* - 1\} \text{ such that } t - rT \in I_2. \end{aligned}$$

For $j = 1, 2$, consider the orthogonal Jacobi field W^j along γ_j defined by:

$$(4.7) \quad W_t^1 = \sum_{r=0}^{k_*} V_{t+rT}, \quad W_t^2 = \sum_{r=0}^{k_*-1} V_{t+rT}.$$

It is not the case that both W^1 and W^2 are everywhere parallel to $\dot{\gamma}_0$ on I_1 and I_2 respectively, for otherwise from (4.7) one would conclude easily that V would be everywhere parallel to $\dot{\gamma}_0$ (Lemma 2.5). Assume that, say, W^1 is not everywhere parallel to $\dot{\gamma}_0$ on I_1 , i.e., by Lemma 2.5, there are only isolated values of t where W_t^1 is parallel to $\dot{\gamma}_0(t)$; the other case is totally analogous. By reducing the size of I_1 , we can assume that W_t^1 is never a multiple of $\dot{\gamma}_0(t)$ on I_1 . Now, the first part of the proof can be repeated, by replacing the Jacobi field V with W^1 . We can find a globally defined symmetric $(0, 2)$ -tensor h on M , with compact support contained in U_1 , with prescribed value H and covariant derivative K in the direction W^1 along $\gamma_0|_{I_1}$. Choose H and K as above, and compute:

$$\begin{aligned} \int_0^1 h(\dot{\gamma}_0, \mathbf{D}V) + \frac{1}{2} \nabla h(V, \dot{\gamma}_0, \dot{\gamma}_0) dt &= \frac{1}{2} \sum_{r=0}^{k_*} \int_{a_1+rT}^{b_1+rT} \nabla h(V, \dot{\gamma}_0, \dot{\gamma}_0) dt \\ &= \frac{1}{2} \int_{a_1}^{b_1} \nabla h(W^1, \dot{\gamma}_0, \dot{\gamma}_0) dt = \frac{1}{2} \int_I K_t(\dot{\gamma}_0(t), \dot{\gamma}_0(t)) dt > 0. \end{aligned}$$

This concludes the proof. \square

4.3. Perturbations of a metric in its conformal class. It is a natural question to ask whether the genericity result of Proposition 4.3 remains true if one consider more restrictive classes of variations of a given metric. Particularly interesting examples are perturbations inside a given conformal class of semi-Riemannian metrics. However, one cannot expect that the genericity result holds in this case, as the following example shows.

2. Example. Let (M, g_0) be a semi-Riemannian manifold, and let $\gamma : [0, 1] \rightarrow M$ be a lightlike geodesic in M with $p = \gamma(0)$ and $q = \gamma(1)$ conjugate along γ . Then, given any semi-Riemannian metric g on M which is conformal to g_0 , there exists a suitable

⁷Here the assumption that $p \neq q$ is being used. Note that if $p = q$, then $t_* = T$, and the argument below fails.

reparameterization $\tilde{\gamma}$ of γ which is a lightlike g -geodesic, and such that p and q are conjugate⁸ along $\tilde{\gamma}$ (see for instance [23, Theorem 2.36]). Thus, conformal perturbations do not destroy degeneracy of lightlike geodesics.

We will show that, *apart from the lightlike case*, generic conformal perturbations are sufficient to destroy degeneracy. In view of Example 2, this is the best possible result.

Given a semi-Riemannian metric tensor \bar{g} on M of class C^k , $k \geq 2$, let us denote by $\mathfrak{C}^k(\bar{g})$ the set of all semi-Riemannian metrics on M that are globally conformal to \bar{g} , i.e., the set of metrics of the form $g = \psi \cdot \bar{g}$ for some function $\psi : M \rightarrow \mathbb{R}^+$ of class C^k . As above, when M is not compact, there is no natural topological structure on $\mathfrak{C}^k(\bar{g})$ that makes it homeomorphic to an open subset of a Banach space. Let us denote by $C^k(M)$ the vector space of all real valued C^k -functions on M . In analogy with the notion of C^k -Whitney type Banach spaces of tensor fields, let us call a C^k -Whitney type Banach space of functions on M a vector subspace \mathcal{F} of $C^k(M)$ endowed with a Banach space norm $\|\cdot\|_{\mathcal{F}}$ satisfying:

- (a) \mathcal{F} contains all the functions in $C^k(M)$ having compact support;
- (b) $\|\cdot\|_{\mathcal{F}}$ -convergence implies C^k -convergence on compact subsets of M .

For instance, given a complete Riemannian metric $g_{\mathbb{R}}$ on M , a C^k -Whitney type Banach space of functions on M can be obtained by setting $\mathcal{F} = \mathfrak{C}^k(M; g_{\mathbb{R}})$, which consists of all functions in $C^k(M)$ that have $g_{\mathbb{R}}$ -bounded derivatives up to order k .

Given a C^k -Whitney type Banach space \mathcal{F} of functions on M and a semi-Riemannian metric tensor \bar{g} on M , let us denote by $\mathfrak{C}^k(\bar{g}; \mathcal{F})$ the set:

$$\mathfrak{C}^k(\bar{g}; \mathcal{F}) = \{\psi \cdot \bar{g} : \psi \in \mathcal{F}\}.$$

and by $\mathfrak{C}_+^k(\bar{g}; \mathcal{F})$ the \mathcal{F} -conformal class of \bar{g} , defined by:

$$\mathfrak{C}_+^k(\bar{g}; \mathcal{F}) = \{\psi \cdot \bar{g} : \psi \in \mathcal{F}, \psi > 0\}.$$

The map $\psi \mapsto \psi \cdot \bar{g}$ gives an identification of the set $\mathfrak{C}^k(\bar{g}; \mathcal{F})$ with the Banach space \mathcal{F} (and of $\mathfrak{C}_+^k(\bar{g}; \mathcal{F})$ with the subset \mathcal{F}_+ of everywhere positive functions of \mathcal{F}); $\mathfrak{C}^k(\bar{g}; \mathcal{F})$ will be thought as a metric space with the induced norm.

Proposition 4.4. *Let M be a smooth manifold, \bar{g} a semi-Riemannian metric tensor on M of class C^k , $k \geq 2$, and let $p, q \in M$ be fixed distinct points. Let $\mathcal{F} \subset C^k(M)$ be a C^k -Whitney type Banach space of functions on M , and let \mathcal{A} be a (non empty) open subset of $\mathfrak{C}^k(\bar{g}; \mathcal{F})$ contained in $\mathfrak{C}_+^k(\bar{g}; \mathcal{F})$. Then, the set of metrics $g \in \mathcal{A}$ such that every nonlightlike g -geodesic in M joining p and q is nondegenerate is generic in \mathcal{A} .*

Proof. Let $g_0 \in \mathcal{A}$ and γ_0 be a non lightlike, i.e., $g_0(\dot{\gamma}_0, \dot{\gamma}_0) \neq 0$, g_0 -geodesic in M joining p and q ; let V be a nontrivial g_0 -Jacobi field along γ_0 that vanishes at both endpoints. We will find a variation h of the form $\psi \cdot g_0$, with $\psi : M \rightarrow \mathbb{R}$ a smooth nonnegative function with *small* compact support, and for which the last term in (4.4) does not vanish. For such a variation h , the last term of (4.4) is easily computed by choosing ∇ to be the Levi-Civita connection of g_0 . Namely, in this case $g_0(\dot{\gamma}_0, \mathbf{D}V)$ vanishes identically; this is because the function $g_0(\dot{\gamma}_0, V)$ is affine, and since it vanishes at 0 and at 1, it must be identically zero, as well as its derivative $g_0(\dot{\gamma}_0, \mathbf{D}V)$. Thus, for such a variation h , the quantity $h(\dot{\gamma}_0, \mathbf{D}V)$ vanishes identically. Moreover, since $\nabla g_0 = 0$, then $\nabla h(V, \dot{\gamma}_0, \dot{\gamma}_0) =$

⁸In the Lorentzian case, conjugate points along lightlike geodesics are preserved even by maps more general than conformal diffeomorphisms. It is not hard to prove (for instance, via bifurcation theory using [19, Corollary 11]) the following:

Lemma. *Let (M_i, g_i) , $i = 1, 2$, be Lorentzian manifolds, and let $\Psi : M_1 \rightarrow M_2$ be a continuous injective map that carries timelike curves to timelike curves and lightlike pre-geodesic to lightlike pre-geodesics. Then, Ψ carries pairs of conjugate points along lightlike geodesics into pairs of conjugate points along lightlike geodesics.*

Note that if Ψ as in the statement of the Lemma is a diffeomorphism, then necessarily Ψ is conformal, by a well known result of Dajczer and Nomizu, see [13].

$V(\psi) \cdot g_0(\dot{\gamma}_0, \dot{\gamma}_0)$. Since we are assuming that the constant $g_0(\dot{\gamma}_0, \dot{\gamma}_0)$ is not null, we have now reduced the problem to determining a smooth nonnegative function ψ with the property that $\int_0^1 V(\psi(\gamma_0(t))) dt \neq 0$; we want such a function ψ with compact support in M . For the construction of such ψ , the procedure is analogous to that in the proof of Proposition 4.3, using Lemma 2.4. Assume first that γ_0 is not a portion of a closed geodesic in M with minimal period $T < 1$. Then, by Lemma 4.2, γ_0 has at most a finite number of self-intersections, and we can find an open subinterval $I \subset [0, 1]$ satisfying properties (a) and (b) in the proof of Proposition 4.3, and an open subset $U \subset M$ containing $\gamma_0(I)$ and such that (4.5) holds. Now, choose a smooth function $\alpha : I \rightarrow \mathbb{R}$ having compact support and such that $\int_I \alpha(t) dt > 0$. By Lemma 2.4 (applied to the case of the trivial vector bundle \mathcal{E} over M whose fiber is one dimensional), we can find a smooth map $\psi : M \rightarrow \mathbb{R}$ having compact support contained in U , such that $\psi(\gamma_0(t)) = 1$ and $V_t(\psi) = \alpha(t)$ for all $t \in I$. With this choice, we have:

$$(4.8) \quad \int_0^1 V_t(\psi) dt \stackrel{\text{by (4.5)}}{=} \int_I V_t(\psi) dt = \int_I \alpha(t) dt > 0.$$

This concludes the proof in the case that γ_0 is not a portion of a closed geodesic. When γ_0 is periodic with period $T < 1$, the construction is totally analogous to the proof of Proposition 4.3. One defines Jacobi fields W^1 and W^2 as in (4.7), open intervals $I_i \subset [0, 1]$ and open subsets $U_i \subset M$ satisfying (4.6); by the same arguments, one obtains that at least one of two Jacobi fields, say W^i , is never parallel to $\dot{\gamma}_0(t)$ on I_i . Define $\psi : M \rightarrow \mathbb{R}$ as above replacing the Jacobi field V with W^i and the interval I with I_i using a smooth function $\alpha : I_i \rightarrow \mathbb{R}$ with compact support and satisfying $\int_{I_i} \alpha(t) dt > 0$. As above, set $h = \psi \cdot g_0$; now, (4.8) is replaced by:

$$\int_0^1 V_t(\psi) dt = \sum_{r=0}^{k_*-i+1} \int_{I_i} V_{t+rT}(\psi) dt = \int_{I_i} W_t^i(\psi) dt = \int_{I_i} \alpha(t) dt > 0. \quad \square$$

4.4. Orthogonally split metrics. Let us now take a product manifold $M = M_1 \times M_2$, with $\dim(M_i) = n_i$, $i = 1, 2$, and consider the subset $\text{Met}_{\text{split}}^k(M_1, M_2)$ of $\text{Met}_{n_2}^k(M)$ consisting of all symmetric $(0, 2)$ -tensors g of class C^k on M such that:

- (a) $g_{(x,y)}((v_1, 0), (0, v_2)) = 0$;
- (b) $g_{(x,y)}$ is positive definite on $T_x M_1 \times \{0\}$;
- (c) $g_{(x,y)}$ is negative definite⁹ on $\{0\} \times T_y M_2$,

for all $(x, y) \in M_1 \times M_2$, all $v_1 \in T_x M_1$ and all $v_2 \in T_y M_2$. Elements of $\text{Met}_{\text{split}}^k(M_1, M_2)$ will be called *orthogonally split* semi-Riemannian metric tensors on $M_1 \times M_2$. More generally, a $(0, 2)$ -tensor field \mathfrak{b} on M will be called *orthogonally split* if it satisfies

$$\mathfrak{b}_{(x,y)}((v_1, 0), (0, v_2)) = 0$$

for all $(x, y) \in M_1 \times M_2$, all $v_1 \in T_x M_1$ and all $v_2 \in T_y M_2$.

Let $\mathcal{E} \subset \mathbf{\Gamma}_{\text{sym}}^k(TM^* \otimes TM^*)$ be a C^k -Whitney type Banach space of tensors on M ; we will denote by $\text{Met}_{\text{split}}^k(M_1, M_2; \mathcal{E})$ the intersection $\text{Met}_{\text{split}}^k(M_1, M_2) \cap \mathcal{E}$. Note that the set $\mathcal{E}_{\text{split}}$ consisting of all orthogonally split tensor fields in \mathcal{E} is a (non trivial) closed subspace of \mathcal{E} . Non triviality follows from the fact that $\mathcal{E}_{\text{split}}$ contains all the orthogonally split tensor fields on M having compact support.

Proposition 4.5. *Let M_1 and M_2 be smooth manifolds, let \mathcal{E} be a C^k -Whitney type Banach space of tensors on the product $M = M_1 \times M_2$, and let \mathcal{A} be an open subset of $\mathcal{E}_{\text{split}}$ with $\mathcal{A} \subset \text{Met}_{\text{split}}^k(M_1, M_2; \mathcal{E})$. Given any two distinct points $p, q \in M$, then the set of all $g \in \mathcal{A}$ such that all g -geodesics in M joining p and q are nondegenerate is generic in \mathcal{A} .*

⁹In fact, rather than (b) and (c), we will use the weaker assumptions that g is nondegenerate on $TM_1 \times \{0\}$ and on $\{0\} \times TM_2$.

Proof. Let $g_0 \in \mathcal{A}$ be fixed and consider a g_0 -geodesic $\gamma_0 = (x_1, x_2)$ joining p and q , and a nontrivial g_0 -Jacobi field $V = (V_1, V_2)$ along γ_0 which vanishes at the endpoints. The proof goes along the same lines as the proof of Proposition 4.3, with the difference that here the variation h has to be found in the Banach space $\mathcal{E}_{\text{split}}$. Again, we will determine the variation h to be an orthogonally split symmetric $(0, 2)$ -tensor field having compact support in M . One has to repeat the proof of Proposition 4.3, which involves the construction of a family of bilinear forms K_t on $T_{\gamma_0(t)}M = T_{x_1(t)}M_1 \oplus T_{x_2(t)}M_2$ with the property that $\int_I K_t(\dot{\gamma}_0(t), \dot{\gamma}_0(t)) dt > 0$ on some given interval I . Recall that in the proof of Proposition 4.3 we are choosing the family H_t to vanish identically. In the case under consideration, the desired K_t can be chosen such that $K_t((v_1, 0), (0, v_2)) = 0$ for every $v_1 \in T_{x_1(t)}M_1$, $v_2 \in T_{x_2(t)}M_2$ and every $t \in I$. Namely, it suffices to choose families of symmetric bilinear forms K_t^i on $T_{x_i(t)}M_i$, $i = 1, 2$, satisfying

$$(4.9) \quad \sum_{i=1}^2 \int_I K_t^i(\dot{x}_i(t), \dot{x}_i(t)) dt > 0$$

and set $K_t((v_1, v_2), (w_1, w_2)) = K_t^1(v_1, w_1) + K_t^2(v_2, w_2)$ for all t . The existence of families K_t^i that satisfy (4.9) is easily proven, keeping in mind that $\dot{x}_1(t)$ and $\dot{x}_2(t)$ are not both zero anywhere. Now, Lemma 2.4 is applied to the vector bundle E over M whose sections are the symmetric $(0, 2)$ -tensors h on M satisfying $h_{(x_1, x_2)}((v_1, 0), (0, v_2)) = 0$ for all $x_i \in M_i$ and all $v_i \in T_{x_i}M_i$. In order to make the result of Lemma 2.4 compatible with formula (4.4), one more detail needs to be clarified. Namely, one needs to consider a connection ∇ in E which is *inherited* from a connection $\tilde{\nabla}$ in TM ; more precisely, ∇ has to be given as the restriction to the subbundle E of the induced connection $\tilde{\nabla}$ on $TM^* \otimes TM^*$. It will not be the case in general that connections on $TM^* \otimes TM^*$ restrict to E , i.e., that covariant derivatives of sections of E remain in E . In order to make the connection $\tilde{\nabla}$ restrictable to E , the corresponding connection $\tilde{\nabla}$ on TM has to be chosen of the form:

$$\tilde{\nabla} = \pi_1^*(\nabla^1) \oplus \pi_2^*(\nabla^2),$$

where ∇^i is a connection on TM_i , and $\pi_i : M_1 \times M_2 \rightarrow M_i$ is the projection, $i = 1, 2$. This concludes the argument. \square

4.5. Globally hyperbolic Lorentzian metrics. Let us now study the nondegeneracy problem for geodesics in *globally hyperbolic* Lorentzian manifolds. A time oriented Lorentzian metric g on a connected manifold M is said to be globally hyperbolic if (M, g) admits a *Cauchy surface* Σ , i.e., Σ is a spacelike hypersurface of M which is met exactly once by every non extendible causal curve. There are several equivalent notions of global hyperbolicity that will not be discussed here (see [7, 10, 24] for details). Let us recall that by a classical result by Geroch [15], whose statement has been recently strengthened by Bernal and Sánchez in [8, 9], a globally hyperbolic Lorentzian manifold (M, g) is isometric to a product $\Sigma \times \mathbb{R}$, where Σ is any Cauchy surface of (M, g) , endowed with an orthogonally split metric tensor which is positive definite on the factor Σ and negative definite on the one-dimensional factor \mathbb{R} . We will then consider a manifold M of the form $\Sigma \times \mathbb{R}$, where Σ is a smooth manifold endowed with a complete Riemannian metric g^0 ; we will denote by $\pi_\Sigma : \Sigma \times \mathbb{R} \rightarrow \Sigma$ the projection onto the first factor. We will study the set of metrics $g^{\alpha, \beta}$ on M , where:

- α is a fixed smooth section of the pull-back bundle $\pi_\Sigma^*(T\Sigma^* \otimes T\Sigma)$ such that $g_x^0(\alpha_{(x,s)}, \cdot)$ is positive definite on $T_x\Sigma$ for all $x \in \Sigma$ and all $s \in \mathbb{R}$;
- $\beta : \Sigma \times \mathbb{R} \rightarrow \mathbb{R}^+$ is a smooth positive function,

and the metric tensor $g^{\alpha, \beta}$ is defined by:

$$(4.10) \quad g_{(x,s)}^{\alpha, \beta}((v, r), (w, \bar{r})) = g_x^0(\alpha_{(x,s)}v, w) - \beta_{(x,s)}r\bar{r},$$

for all $x \in \Sigma$, $s \in \mathbb{R}$, $v, w \in T_x \Sigma$, $r, \bar{r} \in T_t \mathbb{R} \cong \mathbb{R}$. A genericity result totally analogous to Proposition 4.5 holds for the family of metrics $g^{\alpha, \beta}$, that can be described simply as metric of splitting type on a product manifold $M_1 \times M_2$ with M_2 one-dimensional. We will be interested in studying the genericity of nondegeneracy property in the subfamily of the $g^{\alpha, \beta}$ consisting of globally hyperbolic metrics.

Given α as above, set:

$$\lambda_{(x,s)}(\alpha) = \|\alpha_{(x,s)}^{-1}\|^{-1},$$

where $\|\cdot\|$ denotes the operator norm on $\text{End}(T_x \Sigma)$ induced by the positive definite inner product g_x^0 . Equivalently, $\lambda_{(x,s)}(\alpha)$ can be defined as the minimum eigenvalue of the positive operator $\alpha_{(x,s)}$ on $T_x \Sigma$. Sufficient conditions for the global hyperbolicity of the Lorentzian metric $g^{\alpha, \beta}$ have been studied in the literature, see [25]; we will be interested in the following:

Proposition 4.6. *Let x_0 be any fixed point in Σ , and denote by $d_0 : \Sigma \rightarrow [0, +\infty[$ be the distance from x_0 function induced by the Riemannian metric g^0 . Assume that for all integer $n > 0$ the following holds:*

$$\sup_{\substack{x \in \Sigma \\ |s| \leq n}} \sqrt{\frac{\beta_{(x,s)}}{\lambda_{(x,s)}(\alpha)(1 + d_0(x)^2)}} < +\infty.$$

Then, for all $s_0 \in \mathbb{R}$, $\Sigma \times \{s_0\}$ is a Cauchy surface of $g^{\alpha, \beta}$. In particular, if Σ is compact then $g^{\alpha, \beta}$ is always globally hyperbolic.

Proof. See [25, Proposition 3.2]. □

Motivated by the result above, let us consider the Banach space \mathcal{G} whose points are pairs (α, β) , where:

- α is a section of class C^2 of the vector bundle $\pi_\Sigma^*(T\Sigma^* \otimes T\Sigma)$ such that $\alpha_{(x,s)}$ is a g_0 -symmetric operator on $T_x \Sigma$ for all (x, s) ;
- $\beta : \Sigma \times \mathbb{R} \rightarrow \mathbb{R}$ is a map of class C^2 ;
- α satisfies the following boundedness assumptions:
 - ◊ $C_0(\alpha) = \sup_{(x,s) \in \Sigma \times \mathbb{R}} \|\alpha_{(x,s)}(1 + d_0(x)^2)\| < +\infty$. Here, $\|\cdot\|$ is the operator norm on $T_x \Sigma$ induced by the Riemannian metric g_0 .
 - ◊ $C_1(\alpha) = \sup_{(x,s) \in \Sigma \times \mathbb{R}} \|\nabla \alpha_{(x,s)}\| < +\infty$. Here, ∇ is the connection on the vector bundle $T^*(\Sigma \times \mathbb{R}) \otimes \pi_\Sigma^*(T\Sigma^* \otimes T\Sigma)$ induced by the Levi-Civita connection of g_0 and the standard connection on the factor \mathbb{R} .
 - ◊ $C_2(\alpha) = \sup_{(x,s) \in \Sigma \times \mathbb{R}} \|\nabla^2 \alpha_{(x,s)}\| < +\infty$. Here, the second covariant derivative of α is taken relatively to the connection on the vector bundle $T^*(\Sigma \times \mathbb{R}) \otimes T^*(\Sigma \times \mathbb{R}) \otimes \pi_\Sigma^*(T\Sigma^* \otimes T\Sigma)$ induced by the Levi-Civita connection of g_0 and the standard connection on the factor \mathbb{R} .
- β satisfies the following boundedness assumptions:
 - ◊ $D_0(\beta) = \sup_{(x,s) \in \Sigma \times \mathbb{R}} |\beta_{(x,s)}| < +\infty$.
 - ◊ $D_1(\beta) = \sup_{(x,s) \in \Sigma \times \mathbb{R}} \|d\beta_{(x,s)}\| < +\infty$.
 - ◊ $D_2(\beta) = \sup_{(x,s) \in \Sigma \times \mathbb{R}} \|\nabla d\beta_{(x,s)}\| < +\infty$. Here, ∇ denotes the covariant derivative of the connection in $T^*(\Sigma \times \mathbb{R})$ induced by the Levi-Civita connection of g_0 and the standard connection on the factor \mathbb{R} .

A Banach space norm on \mathcal{G} is given by:

$$\|(\alpha, \beta)\| = \max \{C_0(\alpha), C_1(\alpha), C_2(\alpha), D_0(\beta), D_1(\beta), D_2(\beta)\}.$$

Proposition 4.7. *Let ε and b be fixed positive real numbers. The subset $\mathcal{A}_{\varepsilon,b} \subset \mathcal{G}$ given by:*

$$\mathcal{A}_{\varepsilon,b} = \left\{ (\alpha, \beta) \in \mathcal{G} : g_0(\alpha_{(x,s)} \cdot, \cdot) \text{ is positive definite, } \inf_{(x,s) \in \Sigma \times \mathbb{R}} \beta_{(x,s)} > 0 \right. \\ \left. \sup_{(x,s) \in \Sigma \times \mathbb{R}} \beta_{(x,s)} < b, \text{ and } \inf_{(x,s) \in \Sigma \times \mathbb{R}} \lambda_{(x,s)}(\alpha)((1 + d_0(x)^2) > \varepsilon \right\}$$

is open in \mathcal{G} . For all $(\alpha, \beta) \in \mathcal{A}_{\varepsilon,b}$, the tensor $g^{\alpha,\beta}$ defined in (4.10) is a globally hyperbolic Lorentzian metric on $\Sigma \times \mathbb{R}$.

Proof. As to the openness of $\mathcal{A}_{\varepsilon,b}$, the only non trivial question is establishing that the assumption

- $g_0(\alpha_{(x,s)} \cdot, \cdot)$ is positive definite
- $\inf_{(x,s) \in \Sigma \times \mathbb{R}} \lambda_{(x,s)}(\alpha)((1 + d_0(x)^2) > \varepsilon$

is open in the topology of \mathcal{G} . This follows immediately from the choice of the semi-norm $C_0(\alpha)$ above, and the fact that the “least eigenvalue function” $T \mapsto \lambda_{\min}(T) \in \mathbb{R}^+$ is Lipschitz with Lipschitzian constant 1 in the set of positive symmetric operators T on a vector space with inner product, that is, $|\lambda_{\min}(T) - \lambda_{\min}(\tilde{T})| \leq \|T - \tilde{T}\|$ (see also footnote (4)).

For $(\alpha, \beta) \in \mathcal{A}_{\varepsilon,b}$, the following inequality holds:

$$(4.11) \quad \sup_{\substack{x \in \Sigma \\ s \in \mathbb{R}}} \sqrt{\frac{\beta_{(x,s)}}{\lambda_{(x,s)}(\alpha)(1 + d_0(x)^2)}} < \sqrt{\frac{b}{\varepsilon}} < +\infty,$$

and the global hyperbolicity of $g^{\alpha,\beta}$ is deduced from Proposition 4.6. \square

Proposition 4.8. *Let p and q be distinct points in $\Sigma \times \mathbb{R}$. For all $\varepsilon, b > 0$, the set of pairs $(\alpha, \beta) \in \mathcal{A}_{\varepsilon,b}$ such that p and q are not conjugate along any $g^{\alpha,\beta}$ -geodesic in $\Sigma \times \mathbb{R}$ is generic in $\mathcal{A}_{\varepsilon,b}$. The open set:*

$$\mathcal{A} = \left\{ (\alpha, \beta) \in \mathcal{G} : g_0(\alpha_{(x,s)} \cdot, \cdot) \text{ is positive definite, } \inf_{(x,s) \in \Sigma \times \mathbb{R}} \beta_{(x,s)} > 0 \right. \\ \left. \sup_{(x,s) \in \Sigma \times \mathbb{R}} \beta_{(x,s)} < +\infty, \text{ and } \inf_{(x,s) \in \Sigma \times \mathbb{R}} \lambda_{(x,s)}(\alpha)((1 + d_0(x)^2) > 0 \right\}$$

contains a dense G_δ consisting of pairs (α, β) such that p and q are nonconjugate along any $g^{\alpha,\beta}$ -geodesic.

Proof. The first statement follows from Proposition 4.5, observing that the vector space $\mathcal{E} = \{g^{\alpha,\beta} : (\alpha, \beta) \in \mathcal{G}\}$ inherits from \mathcal{G} a Banach space norm that makes it into a C^2 -Whitney type Banach space of orthogonally split tensors over $\Sigma \times \mathbb{R}$. Note that \mathcal{G} contains all pairs (α, β) of class C^2 having compact support, and its topology is finer than the weak Whitney C^2 -topology. As to the second statement, it is enough to observe that \mathcal{A} can be described as the countable union $\bigcup_{n \geq 1} \mathcal{A}_{\frac{1}{n}, n}$ of open sets each of which contains a dense G_δ with the desired property. \square

4.6. Stationary Lorentzian metrics. Let us now consider the case of Lorentzian metrics admitting a timelike Killing vector field; we will exhibit an example showing that the transversality condition discussed in Subsection 4.2 does not hold in general in this class.

Let (M, g) be a Lorentzian manifold, and assume the existence of a Killing vector field Y on M . It is a simple observation that an integral line γ of Y is a geodesic in (M, g) if and only if at some point $\gamma(t_0)$ of γ the function $g(Y, Y)$ has a critical point. Namely, since $g(Y, Y)$ is invariant by the flow of Y , the existence of one critical point of $g(Y, Y)$ along γ is equivalent to the fact that every point of γ is critical for $g(Y, Y)$. Now, γ is a geodesic if and only if $\nabla_Y Y = 0$ along γ , i.e., if $g(\nabla_{Y(\gamma(t))} Y, v) = -g(\nabla_v Y, Y) =$

$-\frac{1}{2}v(g(Y, Y)) = 0$ for all t and all $v \in T_{\gamma(t)}M$, i.e., if and only if $\gamma(t)$ is a critical point of $g(Y, Y)$ for all t . The geodesics in (M, g) that are integral lines of Y will be called *vertical*.

Let us show that, given a Lorentzian manifold (M, g) admitting a timelike Killing vector field Y , the transversality condition may fail to hold along vertical geodesics in the class of all Lorentzian metrics on M that have the prescribed field Y as timelike Killing vector field. A stationary Lorentzian manifold (M, g) is said to be *standard* if M is given by a product $M_0 \times \mathbb{R}$, where M_0 is a differentiable manifold, and the metric tensor g is of the form:

$$(4.12) \quad g_{(x,s)}((v, r), (\bar{v}, \bar{r})) = \mathfrak{g}_x(v, \bar{v}) + \mathfrak{g}_x(\delta(x), v)\bar{r} + \mathfrak{g}_x(\delta(x), \bar{v})r - \beta(x)r\bar{r},$$

where $x \in M_0$, $s \in \mathbb{R}$, $v, \bar{v} \in T_x M_0$, $r, \bar{r} \in T_s \mathbb{R} \cong \mathbb{R}$, \mathfrak{g} is a Riemannian metric tensor on M_0 , $\delta \in \mathfrak{X}(M_0)$ is a smooth vector field on M_0 , and $\beta : M_0 \rightarrow \mathbb{R}^+$ is a smooth positive function on M_0 . The field $Y = \partial_s$ tangent to the lines $\{x_0\} \times \mathbb{R}$, $x_0 \in M_0$, is a timelike Killing vector field in (M, g) ; an immediate computation shows that $g_{(x,s)}(Y, Y) = -\beta(x)$ for all $(x, s) \in M_0 \times \mathbb{R}$. Locally, every stationary Lorentzian metric tensor has the form (4.12). When the vector field δ in (4.12) vanishes identically on M_0 , then the metric g is said to be *standard static*.

Let ∇ be the Levi-Civita connection of the metric g in TM_0 ; given a smooth map $f_0 : M_0 \rightarrow \mathbb{R}$, denote by ∇f_0 its gradient relatively to the metric g and by $H^{f_0}(x) : T_x M_0 \rightarrow T_x M_0$, $x \in M_0$, the *Hessian* of f_0 relatively to g at the point x , which is the \mathfrak{g}_x -symmetric linear operator on $T_x M_0$ given by $H^{f_0}(x)v = \nabla_v(\nabla f_0)$, for all $v \in T_x M_0$. If x is a critical point of f_0 , then $\mathfrak{g}_x(H^{f_0}(x)v, w) = d^2 f_0(x)(v, w)$ is the standard second derivative of f_0 at x . A curve $\gamma(t) = (x(t), s(t))$ in M is a geodesic relatively to the metric (4.12) if and only if its components x and s satisfy the system of differential equations:

$$\frac{D}{dt}\dot{x} + \frac{D}{dt}(\dot{s}\delta) - \dot{s}(\nabla\delta)^*(\dot{x}) + \frac{1}{2}\nabla\beta(x)\dot{s}^2 = 0, \quad \frac{d}{dt}[\mathfrak{g}_x(\delta(x), \dot{x}) - \beta(x)\dot{s}] = 0,$$

where $\frac{D}{dt}$ denotes covariant differentiation along x relatively to the connection ∇ , and $(\nabla\delta)^*$ is the $(1, 1)$ -tensor on M defined by $\mathfrak{g}((\nabla\delta)^*(v), w) = \mathfrak{g}(\nabla_w\delta, v)$ for all $v, w \in TM$. As observed above, if x_0 is a critical point of β , i.e., $\nabla\beta(x_0) = 0$, then the curve $\gamma(t) = (x_0, t)$, $t \in [0, 1]$, is a geodesic in (M, g) .

Let us consider for simplicity the static case, i.e., $\delta \equiv 0$. The second variation of the g -geodesic action functional at a given geodesic $\gamma(t) = (x(t), s(t))$, $t \in [0, 1]$, is given by:

$$\begin{aligned} I_{\mathfrak{g},\beta}(\gamma)[(\xi, \sigma), (\bar{\xi}, \bar{\sigma})] &= \int_0^1 \left[\mathfrak{g}\left(\frac{D}{dt}\xi, \frac{D}{dt}\bar{\xi}\right) + \mathfrak{g}(R(\xi, \dot{x})\bar{\xi}, \dot{x}) - \dot{\sigma}'\dot{s}\mathfrak{g}(\nabla\beta(x), \xi) \right. \\ &\quad \left. - \sigma'\dot{s}\mathfrak{g}(\nabla\beta(x), \bar{\xi}) - \frac{1}{2}\dot{s}^2\mathfrak{g}(H^\beta(x)\xi, \bar{\xi}) - \beta(x)\sigma'\bar{\sigma}' \right] dt, \end{aligned}$$

where $\xi, \bar{\xi}$ are variational vector fields along x vanishing at the endpoints, and $\sigma, \bar{\sigma}$ are smooth functions on $[0, 1]$ vanishing at 0 and at 1. In the above formula and in the rest of the section we will denote by a dot the derivatives of the components x and s of the curve γ , and with a prime the derivatives of the component σ of the vector field $V = (\xi, \sigma)$ along γ . A pair (ξ, σ) is a Jacobi field along the geodesic $\gamma = (x, s)$ if it satisfies the second order linear system of differential equations:

$$(4.13) \quad \frac{D^2}{dt^2}\xi - R(\dot{x}, \xi)\dot{x} + \sigma'\dot{s}\nabla\beta(x) + \frac{1}{2}\dot{s}^2 H^\beta(x)\xi = 0,$$

and

$$(4.14) \quad \frac{d}{dt}[\mathfrak{g}(\dot{s}\mathfrak{g}(\nabla\beta(x), \xi) + \beta(x)\sigma')] = 0.$$

In order to construct the required example, let us consider a geodesic of the form $\gamma(t) = (x_0, t)$, $t \in [0, 1]$, where $x_0 \in M_0$ is a critical point of β . Equations (4.13) and (4.14)

become:

$$\frac{D^2}{dt^2} \xi + \frac{1}{2} H^\beta(x_0) \xi = 0, \quad \text{and} \quad \sigma'' = 0.$$

Thus, if $V = (\xi, \sigma)$ is a Jacobi field along γ that vanishes at 0 and at 1, then $\sigma \equiv 0$, while ξ is a smooth curve in $T_{x_0} M_0$ satisfying the first of the two equations above. Note that the covariant derivative $\frac{D}{dt} \xi$ in this case equals the standard derivative ξ' . Assume that this equation has a non trivial solution ξ satisfying $\xi(0) = \xi(1) = 0$ and $\int_0^1 \xi(t) dt = 0$. For instance, one can take $M_0 = \mathbb{R}$, $x_0 = 0$ and $\beta(x) = 1 + 4\pi^2 x^2$; then, $\frac{1}{2} \beta''(0) = 8\pi^2$, and the differential equation $\xi'' + 4\pi^2 \xi = 0$ has the solution $\xi(t) = \sin(2\pi t)$ with the required properties. Similar examples can be given easily in higher dimensions.

An *infinitesimal variation* h of g in the class of stationary metrics on M of the type (4.12) has the form:

$$(4.15) \quad h_{(x,s)}((v,r), (\bar{v}, \bar{r})) = \mathfrak{h}_x(v, \bar{v}) + \mathfrak{g}_x(\rho(x), v) \bar{r} + \mathfrak{g}_x(\rho(x), \bar{v}) r + \zeta(x) r \bar{r},$$

where $x \in M_0$, $s \in \mathbb{R}$, $v, \bar{v} \in T_x M_0$, $r, \bar{r} \in T_s \mathbb{R} \cong \mathbb{R}$, \mathfrak{h} is a symmetric $(0, 2)$ -tensor on M_0 , $\rho \in \mathfrak{X}(M_0)$ is a smooth vector field on M_0 , and $\zeta : M_0 \rightarrow \mathbb{R}$ is a smooth function on M_0 . We claim that for every such h , the quantity $\int_0^1 [h(\dot{\gamma}, \frac{D}{dt} V) + \frac{1}{2} \nabla h(V, \dot{\gamma}, \dot{\gamma})] dt$ vanishes. Namely,

$$h(\dot{\gamma}, \frac{D}{dt} V) = \mathfrak{g}(\rho(x_0), \xi'),$$

and thus

$$\int_0^1 h(\dot{\gamma}, \frac{D}{dt} V) dt = \mathfrak{g}(\rho(x_0), \xi(1) - \xi(0)) = 0.$$

Moreover,

$$\nabla h(V, \dot{\gamma}, \dot{\gamma}) = \nabla_\xi \mathfrak{h}(\dot{x}, \dot{x}) + 2\mathfrak{g}_0(\nabla_\xi \rho, \dot{x}) \dot{s} + \xi(\zeta) \dot{s}^2 = \xi(\zeta);$$

hence:

$$\int_0^1 \nabla h(V, \dot{\gamma}, \dot{\gamma}) dt = \int_0^1 \xi(\zeta) dt = \int_0^1 \mathfrak{g}(\nabla \zeta, \xi) dt = \mathfrak{g}(\nabla \zeta(x_0), \int_0^1 \xi(t) dt) = 0.$$

This proves our claim and gives the desired counterexample in the stationary case.

5. GENERICITY IN THE C^∞ CATEGORY

It is desirable to have a genericity result also in the space of C^∞ -metric tensors, endowed with the Whitney weak C^∞ topology (see for instance [18]). When the base manifold is non compact, the space of all symmetric tensors, endowed with the topology of C^∞ convergence on compact sets, is a only a *Frechet* space, so that our Banach space approach does not apply directly. However, as it was brought to the attention of the authors by the referee, there is an elegant argument due to Taubes that allows to extend to the C^∞ realm our results. The same idea was used in [14], which is where the authors learned about it; we will sketch here the argument adapted to our situation.

Consider a differentiable manifold M , a complete Riemannian metric g_R on M , two distinct points $p, q \in M$, consider the sequence $\mathcal{E}^k = \Gamma_{\text{sym}}^k(TM^* \otimes TM^*; g_R)$ of C^k -Whitney type Banach space of tensor fields on M described in Example 1, Subsection 4.1. Note that the set of tensors of class C^∞ having compact support is dense in each \mathcal{E}^k . In particular, $\mathcal{E}^\infty = \bigcap_{k \geq k_0} \mathcal{E}^k$ is dense in every \mathcal{E}^k . We will think of \mathcal{E}^∞ as a Frechet space endowed with the family of seminorms $\|\cdot\|_k$ defined in (4.1). In particular, \mathcal{E}^∞ is a *Baire space*, i.e., the intersection of a countable family of dense open subsets is dense.

Let $k_0 \geq 2$ be fixed, and let \mathcal{A} be an open subset of \mathcal{E}^{k_0} consisting of nondegenerate tensors, i.e., semi-Riemannian metrics on M . For $k \geq k_0$, set $\mathcal{A}_k = \mathcal{A} \cap \mathcal{E}^k$; this is an open subset of \mathcal{E}^k . Define \mathcal{A}_* to be the subset of \mathcal{A} consisting of all metric tensors for which all geodesics connecting p and q are nondegenerate. By assumption $\mathcal{A}_{k,*} = \mathcal{A}_* \cap \mathcal{A}_k$ is a generic subset of \mathcal{A}_k for all $k \geq k_0$. Finally, define $\mathcal{A}_\infty = \mathcal{A} \cap \mathcal{E}^\infty = \bigcap_{k \geq k_0} \mathcal{A}_k \subset \mathcal{E}^\infty$, which is a dense subset of \mathcal{A}_k for all k , and set $\mathcal{A}_{\infty,*} = \mathcal{A}_* \cap \mathcal{A}_\infty$. Note that \mathcal{A}_∞ is an

open subset of \mathcal{E}^∞ , and thus it is also a Baire space; convergence in \mathcal{A}_∞ implies C^∞ -convergence on compact subsets of M . We want to prove that $\mathcal{A}_{\infty,*}$ is generic in \mathcal{A}_∞ . To this aim, denote by L_R the length functional of curves relative to the Riemannian metric g_R ; for all $M > 0$ define the following sets:

$$\mathcal{A}_{k,*,M} = \{g \in \mathcal{A}_k : \text{all } g\text{-geodesic } \gamma \text{ connecting } p \text{ and } q, \text{ with } L_R(\gamma) \leq M, \text{ are nondegenerate}\},$$

and

$$\mathcal{A}_{\infty,*,M} = \bigcap_{k \geq k_0} \mathcal{A}_{k,*,M}.$$

Clearly, $\mathcal{A}_{\infty,*} = \bigcap_{M=1}^\infty \mathcal{A}_{\infty,*,M}$, thus, to prove our claim it suffices to show that $\mathcal{A}_{\infty,*,M}$ is open and dense in \mathcal{A}_∞ . The key observation is that for all k and M , $\mathcal{A}_{k,*,M}$ is open in \mathcal{A}_k . This follows from the following argument. Assume that $g_n \in \mathcal{A}_k \setminus \mathcal{A}_{k,*,M}$ is a sequence converging to some $g_\infty \in \mathcal{A}_k$. Then, there exists a sequence $\gamma_n : [0, 1] \rightarrow M$ of g_n -geodesics connecting p and q , with $L_R(\gamma_n) \leq M$ for all n , and such that there is a non trivial g_n -Jacobi field J_n along γ_n with $J_n(0) = J_n(1) = 0$ for all n . Each J_n can be normalized so that

$$(5.1) \quad \left\| \frac{D^{g_n}}{dt} J_n(0) \right\| = 1$$

for all n ; here $\frac{D^{g_n}}{dt} J_n$ is the covariant derivative of J_n along γ_n relatively to the Levi-Civita connection of g_n . Using the completeness of g_R , by the theorem of Arzelà and Ascoli, we can assume that the sequence γ_n converges to a curve γ_∞ connecting p and q ; an immediate continuity argument shows that γ_∞ is a g_∞ -geodesic with $L_R(\gamma_\infty) \leq M$. By (5.1), we can also assume that the sequence $v_n = \frac{D^{g_n}}{dt} J_n(0) \in T_p M$ is convergent to some $v_\infty \neq 0$. By continuity, the g_∞ -Jacobi field J_∞ along γ_∞ satisfying $J_\infty(0) = 0$ and $\frac{D^{g_\infty}}{dt} J_\infty(0) = v_\infty$ also satisfies $J_\infty(1) = 0$, i.e., γ_∞ is a degenerate g_∞ -geodesic connecting p and q , and $g_\infty \notin \mathcal{A}_{k,*,M}$. This shows that $\mathcal{A}_{k,*,M}$ is open in \mathcal{A}_k for every $M > 0$ and $k \in \mathbb{N} \cup \{+\infty\}$. Moreover, since $\mathcal{A}_{k,*,M}$ contains $\mathcal{A}_{k,*}$, then $\mathcal{A}_{k,*,M}$ is also dense in \mathcal{A}_k for all M . Finally, since \mathcal{A}_∞ is dense in \mathcal{A}_k and $\mathcal{A}_{k,*,M}$ is open and dense in \mathcal{A}_k , then $\mathcal{A}_\infty \cap \mathcal{A}_{k,*,M} = \mathcal{A}_{\infty,*,M}$ is dense in \mathcal{A}_k for all k , and thus $\mathcal{A}_{\infty,*,M}$ is dense in \mathcal{A}_∞ . This proves the genericity result in the C^∞ -category. Analogous results hold in all the cases discussed in Section 4.

6. A FEW FINAL REMARKS AND OPEN PROBLEMS

Let us conclude with a few observations.

First, one should observe that the genericity result for globally hyperbolic Lorentzian manifolds stated in Subsection 4.5 is far from being conclusive, or exhaustive. Note for instance that Proposition 4.8 does not apply to sets containing metric tensors $g^{\alpha,\beta}$ with β an unbounded function on M . Several different statements of the genericity result are possible by the very same argument, simply by selecting the appropriate set of tensors and its Banach space structure that one wants to consider. It should also be mentioned that somewhat stronger genericity results may be obtained by relaxing the global hyperbolicity condition given in (4.11), in that the inequality may be required to hold in smaller regions of the spacetime. For instance, in [2] it is given a condition on the first derivative of the metric coefficients α and β implying that all the geodesics between the prescribed points p and q remain in a time-limited region of the spacetime. However, such stronger results would certainly have a more involved statement, filled with technicalities that are probably not appropriate for the purposes of the present paper. The interested reader will have no problem in adapting the arguments in the proof of Proposition 4.8 to other specific cases.

As to the stationary Lorentzian case (Subsection 4.6), the *negative* result given by the counterexample exhibited opens several interesting questions and conjectures that deserve

further attention. First, it is natural to conjecture that, apart from vertical geodesics, stationary infinitesimal perturbations of the metric would suffice to destroy degeneracies. Should this be the case, then a genericity result may be obtained by considering points p and q that do not belong to the same integral line of the Killing vector field. A proof for the existence of appropriate infinitesimal perturbations would have to be based on the following conjecture: given a non vertical geodesic $\gamma = (x, s)$ and a nontrivial Jacobi field $J = (\xi, \tau)$ along γ vanishing at the endpoints, then at some instants t , the vector $\xi(t)$ is not parallel to $\dot{x}(t)$. A direct proof of this fact, based on the Jacobi differential equations (4.13) and (4.14), seems to be rather involved, so that a suitable version of Lemma 2.5 would have to be proven. Another interesting point would be to determine the genericity of the nondegeneracy property in the stationary Lorentzian case if one allows that also the Killing vector field Y may be perturbed. We conjecture that the genericity property in this case would hold under no restrictions on the endpoint.

Finally, we would like to mention the case of closed geodesics, which is substantially more involved than the fixed endpoint case. Let us recall that the first statement of the Riemannian bumpy theorem is due to Abraham, see [4], but to the authors' knowledge the first complete proof of it is due to Anosov, see [6]. A very interesting observation is that a similar result does not hold for a general conservative Hamiltonian system, where one can have degenerate periodic orbits that are not destroyed by small perturbations, as shown in [22]. Significant improvements of the bumpy metric theorem have been proven later by Klingenberg and Takens [21], who have shown genericity of the set of metrics with the property that the Poincaré map of every closed geodesic and all its derivatives up to a finite order belong to a prescribed open and dense subset of the space of jets of symplectic maps around a fixed point.

As we have observed, the theory developed in this paper does not work in order to prove a genericity result for closed geodesics: iterates cannot be dealt with the perturbation arguments discussed. Although parts of Anosov's proof of the bumpy metric theorem in [6] can be carried over to the semi-Riemannian case (namely, all the properties depending on the linearized Poincaré map), the positive definite character of Anosov's argument in some parts of the proof cannot be extended directly to the semi-Riemannian case. For instance, it is used in [6] a certain lower bound on the length of closed geodesics for all Riemannian metrics in a neighborhood of a given one; such bound certainly does not exist outside the Riemannian realm. A natural conjecture, or more exactly a wishful thinking at this stage, is that bumpy metrics may be generic in sets of Lorentzian metrics satisfying restrictive causality and geometric assumptions. A natural guess would be starting with the stationary and globally hyperbolic case, where all closed geodesics are spacelike, and recent developments of the variational geodesic theory (refs. [11, 12]) indicate a certain Riemannian behavior of the geodesic flow.

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